Gain of Regularity for the KP-I Equation

Julie Levandosky* Mauricio Sepúlveda† Octavio Vera Villagrán‡

Abstract

In this paper we study the smoothness properties of solutions to the KP-I equation. We show that the equation's dispersive nature leads to a gain in regularity for the solution. In particular, if the initial data ϕ possesses certain regularity and sufficient decay as $x \to \infty$, then the solution $u(t)$ will be smoother than ϕ for $0 < t \leq T$ where T is the existence time of the solution.

Keywords and phrases: KP-I equation, gain in regularity, weighted Sobolev space.

1 Introduction

The KdV equation is a model for water wave propagation in shallow water with weak dispersive and weak nonlinear effects. In 1970, Kadomtsev & Petviashvili [14] derived a two-dimensional analog to the KdV equation. Now known as the KP-I and KP-II equations, these equations are given by

$$u_{tx} + u_{xxxx} + u_{xx} + \epsilon u_{yy} + (uu_x)_x = 0$$

where $\epsilon = \mp 1$. In addition to being used as a model for the evolution of surface waves [1], the KP equation has also been proposed as a model for internal waves in straits or channels of varying depth and width [24], [8]. The KP equation has also been studied as a model for ion-acoustic wave propagation in isotropic media [21]. In this paper we consider smoothness properties of solutions to the KP-I equation

$$(u_t + u_{xxx} + u_x + u\, u_x)_x - u_{yy} = 0, \qquad (x,\, y) \in \mathbb{R}^2, \quad t \in \mathbb{R} \tag{1.1}$$

$$u(x,\, y,\, 0) = \phi(x,\, y). \tag{1.2}$$

Certain results concerning the Cauchy problem for the KP-I equation include the following. Ukai [25] proved local well-posedness for both the KP-I and KP-II equations for initial data in $H^s(\mathbb{R}^2)$, $s \geq 3$, while Saut [23] proved some local existence results for generalized

*Mathematics Department, Framingham State College, Framingham, MA 01701, jlevandosky@frc.mass.edu

†Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile. mauricio@ing-mat.udec.cl

‡Departamento de Matemática, Universidad del Bío-Bío Collao 1202, Casilla 5-C, Concepción, Chile, overa@ubiobio.cl

KP equations. More recently, results concerning global well-posedness for the KP-I equation have appeared. In particular, see the works of Kenig [15] and Molinet, Saut, and Tzvetkov [20]. Here we consider the question of gain of regularity for solutions to the KP-I equation.

A number of results concerning gain of regularity for various nonlinear evolution equations have appeared. This paper uses the ideas of Cohen [4], Kato [13], Craig and Goodman [6] and Craig, Kappeler, and Strauss [7]. Cohen considered the KdV equation, showing that "box-shaped" initial data $\phi \in L^2(\mathbb{R}^2)$ with compact support lead to a solution $u(t)$ which is smooth for $t > 0$. Kato generalized this result, showing that if the initial data ϕ are in $L^2((1 + e^{\sigma x})\, dx)$, the unique solution $u(t) \in C^\infty(\mathbb{R}^2)$ for $t > 0$. Kruzhkov and Faminskii [17] replaced the exponential weight function with a polynomial weight function, quantifying the gain in regularity of the solution in terms of the decay at infinity of the initial data. Craig, Kappeler, and Strauss expanded on the ideas from these earlier papers in their treatment of highly generlized KdV equations.

Other results on gain of regularity for linear and nonlinear dispersive equations include the works of Hayashi, Nakamitsu, and Tsutsumi [10], [11], Hayashi and Ozawa [12], Constantin and Saut [5], Ponce [22], Ginibre and Velo [9], Kenig, Ponce and Vega [16], Vera [26], [27] and Ceballos, Sepulveda and Vera [3].

In studying propagation of singularities, it is natural to consider the bicharacteristics associated with the differential operator. For the KdV equation, it is known that the bicharacteristics all point to the left for $t > 0$, and all singularities travel in that direction. Kato [13] makes use of this uniform dispersion, choosing a nonsymmetric weight function decaying as $x \to -\infty$ and growing as $x \to \infty$. In [7], Craig, Kappeler and Strauss also make use of a unidirectional propagation of singularities in their results on infinite smoothing properties for generalized KdV-type equations for which $f_{u_{xxx}} \geq c > 0$.

For the two-dimensional case, Levandosky [18] proves smoothing properties for the KP-II equation. This result makes use of the fact that the bicharacteristics all point into one half-plane. Subsequently, in [19], Levandosky considers generalized KdV-type equations in two-dimensions, proving that if all bicharacteristics point into one half-plane, an infinite gain in regularity will occur, assuming sufficient decay at infinity of the initial data.

In this paper, we address the question regarding gain in regularity for the KP-I equation. Unlike the KP-II equation, the bicharacteristics for the KP-I equation are not restricted to a half-plane but span all of $\mathbb{R}^2$. As a result, singularities may travel in all of $\mathbb{R}^2$. However, here we prove that if the initial data decays sufficiently as $x \to \infty$, then we will gain a finite number of derivatives in x (as well as mixed derivatives). In order to state a special case of our gain in regularity theorem, we first introduce certain function spaces we will be using.

Definition. We define

$$X^0(\mathbb{R}^2) = \left\{ u : \ u, \ \xi^3 \widehat{u}, \ \frac{\eta^2}{\xi} \widehat{u} \in L^2(\mathbb{R}^2) \right\} \tag{1.3}$$

equipped with the natural norm. On the space

$$\widetilde{X}^0(\mathbb{R}^2) = \left\{ u : \ \frac{1}{\xi} \widehat{u}(\xi, \, \eta) \in L^2(\mathbb{R}^2) \right\} \tag{1.4}$$

we define the operator ∂_x^{-1} by $\widehat{\partial_x^{-1} u} \equiv \frac{1}{i\xi}\,\widehat{u}$. Therefore, in particular, we can write the norm of $X^0(\mathbb{R}^2)$ as

$$||u||^2_{X^0(\mathbb{R}^2)} = \int_{\mathbb{R}^2} [\, u^2 + u^2_{xxx} + (\partial_x^{-1} u_{yy})^2 \,]\, dx\, dy < +\infty \tag{1.5}$$

On this space of functions $X^0(\mathbb{R}^2)$, it makes sense to rewrite (1.1)-(1.2) as

$$u_t + u_{xxx} + u_x + u\, u_x - \partial_x^{-1} u_{yy} = 0, \qquad (x,\, y) \in \mathbb{R}^2, \quad t \in \mathbb{R} \tag{1.6}$$
$$u(x,\, y,\, 0) = \phi(x,\, y) \tag{1.7}$$

and consider weak solutions $u \in X^0(\mathbb{R}^2)$.

Definition. Let N be a positive integer. We define the space of functions $X^N(\mathbb{R}^2)$ as follows

$$X^N = \left\{ u:\ u \in L^2(\mathbb{R}^2),\ \mathcal{F}^{-1}(\xi^3\,\widehat{u}) \in H^N(\mathbb{R}^2),\ \mathcal{F}^{-1}\left(\frac{\eta^2}{\xi}\,\widehat{u}\right) \in H^N(\mathbb{R}^2) \right\} \tag{1.8}$$

equipped with the norm

$$||u||^2_{X^N(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \left(u^2 + \sum_{|\alpha| \le N} [\, (\partial^\alpha u_{xxx})^2 + (\partial^\alpha \partial_x^{-1} u_{yy})^2 \,] \right) dx\, dy < +\infty \tag{1.9}$$

where $\alpha = (\alpha_1,\, \alpha_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $|\alpha| = \alpha_1 + \alpha_2$.

Gain of Regularity Theorem. *Let u be a solution of (1.6)-(1.7) in $\mathbb{R}^2 \times [0, T]$ such that $u \in L^\infty([0,T]; X^1(\mathbb{R}^2))$ and*

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^2} [u^2 + (\partial_y^L u)^2 + (1 + x_+)^L (\partial_x^L u)^2]\, dx\, dy < +\infty \tag{1.10}$$

for some integer $L \ge 2$. Then

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^2} t^{|\alpha| - L} (x_+^{2L - |\alpha| - \alpha_2} + e^{\sigma x_-}) (\partial^\alpha u)^2 + \int_0^T \int_{\mathbb{R}^2} t^{|\alpha| - L} (x_+^{2L - |\alpha| - \alpha_2 - 1} + e^{\sigma x_-}) (\partial^\alpha u_x)^2 < \infty,$$

for $L + 1 \le |\alpha| \le 2L - 1$, $2L - |\alpha| - \alpha_2 \ge 1$, $\sigma > 0$ arbitrary.

Remarks.

(1) If we consider $|\alpha| = 2L - 1$ above, then $\alpha = (2L - 1, 0)$ in which case the result states that

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^2} t^{L-1} (x_+ + e^{\sigma x_-}) (\partial_x^{2L-1} u)^2 + \int_0^T \int_{\mathbb{R}^2} t^{L-1} (1 + e^{\sigma x_-}) (\partial_x^{2L} u)^2 < \infty.$$

In particular, this result shows a *gain* in L derivatives in x.

(2) While we gain x derivatives and mixed derivatives, we do not gain pure y derivatives. However, we do not require any weighted estimates on $\partial_y^L u$. In addition, we do not require any weighted estimates on u. The results on the KP-II equation include gains in pure y derivatives, but also require weighted estimates on $\partial_y^L u$.

(3) The assumptions on u are reasonable and shown to hold in section 6.

The main idea of the proof is the following. We use an inductive argument where on each level $|\alpha|$, we apply the operator $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$ to (1.6), multiply the differentiated equation by $2f_\alpha \partial^\alpha u$ where f_α is our weight function, to be specified later, and integrate over $\mathbb{R}^2$. Doing so, we arrive at the following inequality

$$\begin{aligned}\partial_t \int f(\partial^\alpha u)^2 + 3\int f_x(\partial^\alpha u_x)^2 &\leq \int f_x(\partial^\alpha \partial_x^{-1} u_y)^2 \\ &+ \int [f_t + f_{xxx} + f_x](\partial^\alpha u)^2 + \left| 2\int f(\partial^\alpha u)\partial^\alpha(uu_x)\right|\end{aligned} \tag{1.11}$$

where $\int = \int_{\mathbb{R}^2} dxdy$. Assuming $f_x > 0$, the second term on the left-hand side has a positive sign, thus allowing us to prove a gain in regularity. We notice that the first term on the right-hand side is of order $|\alpha|$. By choosing appropriate weight functions for each α, we have a bound on that term from the previous step of the induction. After proving estimates involving the nonlinear term on the right-hand side of the equation, we apply Gronwall's inequality to prove the bounds on the terms on the left-hand side of the equation.

The plan of the paper is the following. In section 2 we show the derivation of (1.11). In sections 3 and 4 we prove an existence result showing that for initial data $\phi \in X^N(\mathbb{R}^2)$ there exists a smooth solution $u \in L^\infty([0,T]; X^N(\mathbb{R}^2))$ for a time T depending only on $||\phi||_{X^0}$. In section 5 we prove estimates for the terms on the right-hand side of (1.11). In section 6 we prove a priori estimates showing that the solution u found in section 4 also satisfies (1.10) for the same time T as long as

$$\int (\phi^2 + (\partial_y^L \phi)^2 + (1+x_+)^L(\partial_x^L \phi)^2) < \infty.$$

Once we have found the solution u in the appropriate weighted space as well as bounds for terms on the right-hand side of (1.11), in section 7, we can state and prove our main gain in regularity result. This proof uses an inductive argument along with the main estimates proven in section 5.

Choice of weight function. We will be using non-symmetric weight functions. In particular, we will be using weight functions $f(x,t) \in C^\infty$ which behave roughly like powers of x for $x > 1$ and decay exponentially for $x < -1$. We define our weight classes as follows.

Definition. A function $f = f(x,\, t)$ belongs to the weight class $W_{\sigma\, i\, k}$ if it is a positive C^∞ function on $\mathbb{R} \times [0,\, T]$ and there are constants c_j, $0 \leq j \leq 5$ such that

$$0 < c_1 \leq t^{-k}\, e^{-\sigma x}\, f(x,\, t) \leq c_2 \qquad \forall\, x < -1, \quad 0 < t < T. \tag{1.12}$$

$$0 < c_3 \leq t^{-k}\, x^{-i}\, f(x,\, t) \leq c_4 \qquad \forall\, x > 1, \quad 0 < t < T. \tag{1.13}$$

$$(t\mid \partial_t f\mid + \mid \partial_x^r f\mid)\,/f \leq c_5 \qquad \forall\, (x,\, t) \in \mathbb{R}\times[0,\, T], \quad \forall\, r \in \mathbb{N}. \tag{1.14}$$

Thus f looks like t^k as $t \to 0$, like x^i as $x \to +\infty$ and like $e^{\sigma x}$ as $x \to -\infty$.

Before proceeding, we introduce some other function spaces we will be using.

Definition. Let N be a positive integer. Let $\widetilde{H}^N_x(W_{\sigma\, i\, k})$ be the space of functions

$$\widetilde{H}^N_x(W_{\sigma\, i\, k}) = \left\{ v \colon \mathbb{R}^2 \to \mathbb{R} \ : \quad ||v||^2_{\widetilde{H}^N_x(W_{\sigma\, i\, k})} = \int \sum_{|\alpha| \le N} [(\partial^\alpha v)^2 + f|\partial^N_x v|^2] < +\infty \right\} \quad (1.15)$$

with $f \in W_{\sigma\, i\, k}$ fixed.

Remarks.

(1) We note that although the norm above depends on f, all choices of f in this class lead to equivalent norms.

(2) The usual Sobolev space is $H^N(\mathbb{R}^2)$ without a weight.

Definition. For fixed $f \in W_{\sigma\, i\, k}$ define the space (N be a positive integer)

$$\begin{aligned} &L^2([0, T] : \widetilde{H}^N_x(W_{\sigma\, i\, k})) \\ &= \left\{ v(x, y, t) : ||v||^2_{L^2([0, T] : \widetilde{H}^N_x(W_{\sigma\, i\, k}))} = \int_0^T ||v(\cdot, \cdot, t)||^2_{\widetilde{H}^N_x(W_{\sigma\, i\, k})} dt < +\infty \right\} \end{aligned} \quad (1.16)$$

$$\begin{aligned} &L^\infty([0, T] : \widetilde{H}^N_x(W_{\sigma\, i\, k})) \\ &= \left\{ v(x, y, t) : ||v||_{L^\infty([0, T] : \widetilde{H}^N_x(W_{\sigma\, i\, k}))} = \sup_{t \in [0, T]} ||v(\cdot, \cdot, t)||_{\widetilde{H}^N_x(W_{\sigma\, i\, k})} < +\infty \right\} \end{aligned} \quad (1.17)$$

For simplicity, let

$$\mathcal{Z}_L = X^1(\mathbb{R}^2) \bigcap \widetilde{H}^L_x(W_{0\, L\, 0}). \quad (1.18)$$

With this notation, $\mathcal{Z}_L$ consists of those functions u such that

$$||u||^2_{\mathcal{Z}_L} = \int_{\mathbb{R}^2} [u^2 + u^2_{xxxx} + (\partial_x^{-1} u_{yy})^2 + u^2_{yy} + \sum_{|\alpha| \le L} (\partial^\alpha u)^2 + f(\partial^L_x u)^2] \, dx \, dy \quad (1.19)$$

for some $f \in W_{0\, L\, 0}$.

We now state a lemma describing one of the types of bounds we will be using for our a priori estimates.

Lemma 1.1. *For* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} < 1$, $u \in L^2(\mathbb{R}^2)$,

$$||u||_{L^\infty(\mathbb{R}^2)} \le c \left(\int_{\mathbb{R}^2} [\, 1 + |\xi|^p + |\eta|^q \,] \, |\widehat{u}|^2 \, d\xi \, d\eta \right)^{1/2}. \quad (1.20)$$

Proof. The proof follows from writing u in terms of its inverse Fourier transform and using the fact that

$$\int_{\mathbb{R}^2} \frac{1}{1+|\xi|^p+|\eta|^q}\, d\xi\, d\eta < +\infty$$

for p, q satisfying our hypothesis. $\square$

In particular, we have:

$$||u||_{L^\infty(\mathbb{R}^2)} \leq c \left(\int_{\mathbb{R}^2} [\, u^2 + u_{xx}^2 + u_y^2 \,]\, dx\, dy \right)^{1/2}. \tag{1.21}$$

2 Main Equality

We consider the KP-I equation

$$u_t + u_{xxx} + u_x + u\, u_x - \partial_x^{-1} u_{yy} = 0, \qquad (x,\, y) \in \mathbb{R}^2, \quad t \in \mathbb{R} \tag{2.1}$$

$$u(x,\, y,\, 0) = \phi(x,\, y). \tag{2.2}$$

Lemma 2.1. *Let u be a solution of (2.1)-(2.2) with enough Sobolev regularity and with sufficient decay at infinity. Let $f = f(x,t)$. Then*

$$\partial_t \int_{\mathbb{R}^2} f (\partial^\alpha u)^2\, dx\, dy + \int_{\mathbb{R}^2} g\, (\partial^\alpha u_x)^2\, dx\, dy \\ + \int_{\mathbb{R}^2} \theta\, (\partial^\alpha u)^2\, dx\, dy + \int_{\mathbb{R}^2} \theta_1\, (\partial^\alpha \partial_x^{-1} u_y)^2\, dx\, dy + \int_{\mathbb{R}^2} R_\alpha\, dx\, dy = 0 \tag{2.3}$$

such that

$$\begin{aligned} g &= 3\, f_x \\ \theta &= -\, [f_t + f_{xxx} + f_x\,] \\ \theta_1 &= -\, f_x \\ R_\alpha &= 2 \sum_{n=0}^{\alpha_1} \sum_{m=0}^{\alpha_2} \binom{\alpha_1}{n} \binom{\alpha_2}{m} f\, (\partial^\alpha u)\, (\partial_x^n \partial_y^m u)\, (\partial_x^{\alpha_1+1-n} \partial_y^{\alpha_2-m} u). \end{aligned}$$

Proof. Applying the operator ∂^α to (2.1), we have

$$\partial^\alpha u_t + \partial^\alpha u_{xxx} + \partial^\alpha u_x + \partial^\alpha (u\, u_x) - \partial^\alpha \partial_x^{-1} u_{yy} = 0.$$

Multiplying by $2\, f\, \partial^\alpha u$ and integrating over $\mathbb{R}^2$, we have

$$2 \int f\, (\partial^\alpha u)\, (\partial^\alpha u)_t + 2 \int f\, (\partial^\alpha u)\, (\partial^\alpha u_{xxx}) + 2 \int f\, (\partial^\alpha u)\, (\partial^\alpha u_x) \\ + 2 \int f\, (\partial^\alpha u)\, \partial^\alpha (u\, u_x) - \; 2 \int f\, (\partial^\alpha u)\, (\partial^\alpha \partial_x^{-1} u_{yy}) = 0. \tag{2.4}$$

Each term in (2.4) is calculated separately integrating by parts

$$2\int f\,(\partial^\alpha u)\,(\partial^\alpha u)_t = \partial_t \int f\,(\partial^\alpha u)^2 - \int f_t\,(\partial^\alpha u)^2.$$

$$2\int f\,(\partial^\alpha u)\,(\partial^\alpha u_{xxx}) = 3\int f_x\,(\partial^\alpha u_x)^2 - \int f_{xxx}\,(\partial^\alpha u)^2.$$

$$2\int f\,(\partial^\alpha u)\,(\partial^\alpha u_x) = -\int f_x\,(\partial^\alpha u)^2$$

$$-\,2\int f\,(\partial^\alpha u)\,(\partial^\alpha \partial_x^{-1} u_{yy}) = -\int f_x\,(\partial^\alpha \partial_x^{-1} u_y)^2.$$

$$2\int f\,(\partial^\alpha u)\,\partial^\alpha(u\,u_x) = \;2\sum_{n=0}^{\alpha_1}\sum_{m=0}^{\alpha_2}\binom{\alpha_1}{n}\binom{\alpha_2}{m}\int f\,(\partial^\alpha u)\,(\partial_x^n\partial_y^m u)\,(\partial_x^{\alpha_1+1-n}\partial_y^{\alpha_2-m} u).$$

Replacing in (2.4) we obtain

$$\begin{aligned}
&\partial_t \int f\,(\partial^\alpha u)^2 + 3\int f_x\,(\partial^\alpha u_x)^2 \\
&-\int [f_t + f_{xxx} + f_x]\,(\partial^\alpha u)^2 - \int f_x\,(\partial^\alpha \partial_x^{-1} u_y)^2 \\
&+\;2\sum_{n=0}^{\alpha_1}\sum_{m=0}^{\alpha_2}\binom{\alpha_1}{n}\binom{\alpha_2}{m}\int f\,(\partial^\alpha u)\,(\partial_x^n\partial_y^m u)\,(\partial_x^{\alpha_1+1-n}\partial_y^{\alpha_2-m} u) = 0.
\end{aligned}$$

Therefore, we obtain the **Main Equality**,

$$\partial_t \int f(\partial^\alpha u)^2 + 3\int f_x\,(\partial^\alpha u_x)^2 + \int \theta\,(\partial^\alpha u)^2 + \int \theta_1\,(\partial^\alpha \partial_x^{-1} u_y)^2 + \int R_\alpha = 0$$

such that

$$\begin{aligned}
\theta &= -\,[f_t + f_{xxx} + f_x\,] \\
\theta_1 &= -\,f_x \\
R_\alpha &= 2\sum_{n=1}^{\alpha_1}\sum_{m=1}^{\alpha_2}\binom{\alpha_1}{n}\binom{\alpha_2}{m} f\,(\partial^\alpha u)\,(\partial_x^n\partial_y^m u)\,(\partial_x^{\alpha_1+1-n}\partial_y^{\alpha_2-m} u).
\end{aligned}$$

3 An a priori estimate

In section four we prove a basic local-in-time existence theorem for (2.1)-(2.2). The proof relies on approximating (2.1) by a sequence of linear equations. In this section, we prove

an existence theorem for linear equations as well as an a priori estimate on those solutions which will be necessary for our main existence theorem in the next section.

We begin by approximating (2.1) by the linear equation

$$u_t^{(n)} + u_{xxx}^{(n)} + u_x^{(n)} + u^{(n-1)}\, u_x^{(n)} - \partial_x^{-1} u_{yy}^{(n)} = 0 \tag{3.1}$$

where the initial condition is given by $u^{(n)}(x,\, y,\, 0) = \phi(x,\, y)$ and the first approximation is given by $u^{(0)}(x,\, y,\, t) = \phi(x,\, y)$. The linear equation which is to be solved at each iteration is of the form

$$u_t + u_{xxx} + u_x + b\, u_x - \partial_x^{-1} u_{yy} = 0. \tag{3.2}$$

where b is a smooth bounded coefficient. Below we show that this equation can be solved in any interval of time in which the coefficient is defined.

Lemma 3.1 (Existence for linear equation). *Given initial data $\phi \in H^\infty(\mathbb{R}^2) = \bigcap_{N\geq 0} H^N(\mathbb{R}^2)$ and $\partial_x^{-1}\phi_{yy} \in \bigcap_{N\geq 0} H^N(\mathbb{R}^2)$ there exists a unique solution of* (3.2)*. The solution is defined in any time interval in which the coefficients are defined.*

Proof. Let $T > 0$ be arbitrary and $M > 0$ be a constant. Let

$$\mathcal{L} = \partial_t + \partial_x^3 + \partial_x + b\, \partial_x - \partial_x^{-1}\partial_y^2$$

be defined on those functions $u \in X^0(\mathbb{R}^2)$. Recall that $u \in X^0(\mathbb{R}^2)$ means $u,\, u_{xxx},\, \frac{\eta^2}{\xi}\widehat{u} \in L^2(\mathbb{R}^2)$. We consider the bilinear form $\mathcal{B} : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$,

$$\mathcal{B}(u,\, v) = \langle u,\, v\rangle = \int_0^T \int_{\mathbb{R}^2} e^{-M\,t}\, u\, v\, dx\, dy\, dt$$

where $\mathcal{D} = \{u \in C([0,\, T] : L^2(\mathbb{R}^2)) : u(x,\, y,\, 0) = 0\}$. By integration by parts, we see that

$$\begin{aligned} \int_{\mathbb{R}^2} \mathcal{L}u \cdot u\, dx\, dy &= \frac{1}{2}\,\partial_t \int_{\mathbb{R}^2} u^2\, dx\, dy - \frac{1}{2}\int_{\mathbb{R}^2} b_x\, u^2\, dx\, dy \\ &\geq \frac{1}{2}\partial_t \int_{\mathbb{R}^2} u^2\, dx\, dy - \frac{1}{2}\int_{\mathbb{R}^2} cu^2\, dx\, dy \end{aligned}$$

We multiply by $e^{-M\,t}$ and integrate in time to obtain for $u \in C([0,\, T] : X^0(\mathbb{R}^2))$ with $u(x,\, y,\, 0) = 0$

$$\langle \mathcal{L}u,\, u\rangle \geq e^{-M\,t} \int_{\mathbb{R}^2} u^2\, dx\, dy + (M - c) \int_{\mathbb{R}^2} e^{-M\,t}\, u^2\, dx\, dy. \tag{3.3}$$

Thus, $\langle \mathcal{L}u,\, u\rangle \geq \langle u,\, u\rangle$ provided M is chosen large enough. Similarly, $\langle \mathcal{L}^*v,\, v\rangle \geq \langle v,\, v\rangle$ for all $v \in C([0,\, T] : X^0(\mathbb{R}^2))$ such that $v(x,\, y,\, T) = 0$ where $\mathcal{L}^*$ denotes the formal adjoint of $\mathcal{L}$. Therefore, $\langle \mathcal{L}^*v,\, \mathcal{L}^*u\rangle$ is an inner product on $\mathcal{D}^* = \{v \in C([0,\, T] : X^0(\mathbb{R}^2)) : v(x,\, y,\, T) = 0\}$. Denote by Y the completion of $\mathcal{D}^*$ with respect to this inner product. By the Riesz representation theorem, there exists a unique solution $V \in Y$, such that for

any $v \in \mathcal{D}^*$, $\langle \mathcal{L}^* V,\, \mathcal{L}^* v\rangle = (\phi,\, v(x,\, y,\, 0))$ where we used the fact that $(\phi,\, v(x,\, y,\, 0))$ is a bounded linear functional on $\mathcal{D}^*$. Then $w = \mathcal{L}^* V$ is a weak solution of $\mathcal{L} w = 0$, $w_0 = \phi$ with $w \in L^2(\mathbb{R}^2 \times [0,\, T])$.

Remark. To obtain higher regularity of the solution, we repeat the proof with higher derivatives included in the inner product. It is a standard approximation procedure to obtain a result for general initial data. $\square$

Next, we need to introduce a new function space. Let

$$Z_T^N = \{u :\ u \in L^\infty([0,\, T] :\ H^{(N+3,\, N+2)}(\mathbb{R}^2)),\ u_t \in L^\infty([0,\, T] :\ H^N(\mathbb{R}^2))\} \tag{3.4}$$

where $H^{(\alpha_1,\, \alpha_2)}(\mathbb{R}^2) = \{u :\ u,\, \partial_x^{\alpha_1} u,\, \partial_y^{\alpha_2} u \in L^2(\mathbb{R}^2)\}$ with the accompanying norm

$$||u||^2_{Z_T^N} = \sup_{t\in[0,\,T]} \int_{\mathbb{R}^2} \left(u^2 + \sum_{|j|=N} [\, (\partial^j u_{xxx})^2 + (\partial^j u_{yy})^2\,] \right) + \int_{\mathbb{R}^2} \left(u_t^2 + \sum_{|j|=N} (\partial^j u_t)^2 \right). \tag{3.5}$$

Using this function space and the linearized equation (3.1), we consider the mapping $\Pi : Z_T^N \to Z_T^N$ such that $u^{(n)} = \Pi(u^{(n-1)})$ and our first approximation is given by $u^{(0)}(x, y, t) = \phi(x, y)$. In Lemma 3.2 below, we show an a priori estimate which will be used on our sequence of solutions $\{u^{(n)}\}$ in our main existence theorem in section four.

Lemma 3.2. *Let $v,\, w$ be a pair of functions in Z_t^N for all N and all $t \geq 0$, such that $v,\, w$ are solutions to*

$$v_t + v_{xxx} + v_x + w\, v_x - \partial_x^{-1} v_{yy} = 0. \tag{3.6}$$

Then for all $N \geq 0$, the following inequality holds:

$$||v||^2_{Z_t^N} \leq ||v(\,\cdot\,,\,\cdot\,,\,0)||^2_{H^{(N+3,\,N+2)}(\mathbb{R}^2)} + ||v_t(\,\cdot\,,\,\cdot\,,\,0)||^2_{H^N(\mathbb{R}^2)} + c\, t\, ||w||_{Z_t^N}\, ||v||^2_{Z_t^N} \tag{3.7}$$

for all $t \geq 0$.

Proof. We will show that for each j, $|j| \geq 0$ and $0 \leq \widetilde{t} \leq t$,

$$\begin{aligned} \partial_t \int [\, (\partial^j v(\,\cdot\,,\,\cdot\,,\,\widetilde{t}))^2 &+ (\partial^j \partial_x^3 v(\,\cdot\,,\,\cdot\,,\,\widetilde{t}))^2 + (\partial^j \partial_y^2 v(\,\cdot\,,\,\cdot\,,\,\widetilde{t}))^2 + (\partial^j v_t(\,\cdot\,,\,\cdot\,,\,\widetilde{t}))^2\,] \\ &\leq c\, ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}. \end{aligned}$$

We begin by taking j derivatives of (3.6). We have

$$\partial^j v_t + \partial^j v_{xxx} + \partial^j v_x + \partial^j (w\, v_x) - \partial^j \partial_x^{-1} v_{yy} = 0. \tag{3.8}$$

Multiply (3.8) by $2\, \partial^j v$ and integrate over $\mathbb{R}^2$. Hence

$$\begin{aligned} \partial_t \int (\partial^j v(\,\cdot\,,\,\cdot\,,\,\widetilde{t}))^2 \ &\leq\ c \left| \int \partial^j (w\, v_x)\, (\partial^j v) \right| \\ &\leq\ \left| \int [\, (\partial^j w)\, v_x\ +\ \ldots\ +\ w\, (\partial^j v_x)\,]\, (\partial^j v) \right|. \end{aligned}$$

The remainder terms can be bounded as follows:

$$\begin{aligned}
\left|\int (\partial^j w)\, v_x\, (\partial^j v)\right| &\leq \; ||v_x||_{L^\infty(\mathbb{R}^2)} \left(\int (\partial^j w)^2\right)^{1/2} \left(\int (\partial^j v)^2\right)^{1/2} \\
&\leq \; c \left(\int [\, v_x^2 + v_{xxx}^2 + v_{xy}^2\,]\right)^{1/2} ||w||_{H^{|j|}(\mathbb{R}^2)}\, ||v||_{H^{|j|}(\mathbb{R}^2)} \\
&\leq \; c\, ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}
\end{aligned}$$

and

$$\begin{aligned}
\left|\int w\, (\partial^j v_x)\, (\partial^j v)\right| &\leq \; \left|\int w_x\, (\partial^j v)^2\right| \\
&\leq \; c\, ||w_x||_{L^\infty(\mathbb{R}^2)} \int (\partial^j v)^2 \\
&\leq \; c \left(\int [\, w_x^2 + w_{xxx}^2 + w_{xy}^2\,]\right)^{1/2} ||v||^2_{Z_t^{|j|}} \\
&\leq \; c\, ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}.
\end{aligned}$$

Therefore, we obtain

$$\partial_t \int (\partial^j v(\,\cdot\,,\,\cdot\,,\,\widetilde{t}))^2 \leq c\, ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}.$$

Next, we take three x derivatives of (3.8), multiply by $2\,\partial^j v_{xxx}$ and integrate over $\mathbb{R}^2$. Our inequality becomes

$$\begin{aligned}
\partial_t \int \partial^j v_{xxx} &\leq c \left|\int (\partial^j (w\, v_x)_{xxx})\, (\partial^j v_{xxx})\right| \\
&\leq c \left|\int \partial^j (w_{xxx}\, v_x + 2\, w_{xx}\, v_{xx} + 2\, w_x\, v_{xxx} + w\, v_{xxxx})(\partial^j v_{xxx})\right| \\
&\leq c \left|\int \partial^j (w_{xxx}\, v_x)\, (\partial^j v_{xxx})\right| + c \left|\int \partial^j (w_{xx}\, v_{xx})\, (\partial^j v_{xxx})\right| \\
&\qquad + \; c \left|\int \partial^j (w_x\, v_{xxx})\, (\partial^j v_{xxx})\right| + c \left|\int \partial^j (w\, v_{xxxx})\, (\partial^j v_{xxx})\right| \\
&\leq I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We will look at terms $I_k,\ k = 1,\,2,\,3,\,4$ below. For I_1 we have

$$
\begin{aligned}
&\left| \int \partial^j (w_{xxx}\, v_x)\, (\partial^j v_{xxx}) \right| \\
&\quad = \left| \int [\, \partial^j w_{xxx})\, v_x\, (\partial^j v_{xxx}) \;+\; \ldots \;+\; w_{xxx}\, (\partial^j v_x)\, (\partial^j v_{xxx})\,] \right| \\
&\quad \leq ||v_x||_{L^\infty(\mathbb{R}^2)} \left(\int (\partial^j w_{xxx})^2 \right)^{1/2} \left(\int (\partial^j v_{xxx})^2 \right)^{1/2} \\
&\qquad + \cdots + ||\partial^j v_x||_{L^\infty(\mathbb{R}^2)} \left(\int w_{xxx}^2 \right)^{1/2} \left(\int (\partial^j v_{xxx})^2 \right)^{1/2} \\
&\quad \leq ||v||_{Z_t^0}\, ||w||_{Z_t^{|j|}}\, ||v||_{Z_t^{|j|}} + \cdots + ||v||_{Z_t^{|j|}}\, ||w||_{Z_t^0}\, ||v||_{Z_t^{|j|}} \\
&\quad \leq c\, ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}.
\end{aligned}
$$

For I_2,

$$
\left| \int \partial^j (w_{xx}\, v_{xx})\, (\partial^j v_{xxx}) \right| = \left| \int [\, (\partial^j w_{xx})\, v_{xx} + \ \ldots\ + w_{xx} (\partial^j v_{xx})\,]\, (\partial^j v_{xxx}) \right|.
$$

To bound these terms, we will use the following anisotropic imbedding in [2]. For $2 \leq n < 6$,

$$
\left(\int_{\mathbb{R}^2} |u|^n \right)^{1/n} \leq \left(\int_{\mathbb{R}^2} [\, u^2 + u_x^2 + (\partial_x^{-1} u_y)^2\,] \right)^{1/2}. \tag{3.9}
$$

We will look at the most difficult terms to bound below.

$$
\begin{aligned}
&\left| \int (\partial^j w_{xx})\, v_{xx}\, (\partial^j v_{xxx}) \right| \\
&\quad \leq \left(\int (\partial^j w_{xx})^4 \right)^{1/4} \left(\int (v_{xx})^4 \right)^{1/4} \left(\int (\partial^j v_{xxx})^2 \right)^{1/2} \\
&\quad \leq \left(\int [\, (\partial^j\, w_{xx})^2 + (\partial^j\, w_{xxx})^2 + (\partial^j\, w_{xy})^2\,] \right)^{1/2} \\
&\qquad \times \left(\int [\, (v_{xx})^2 + (v_{xxx})^2 + (v_{xy})^2\,] \right)^{1/2} \left(\int (\partial^j v_{xxx})^2 \right)^{1/2} \\
&\quad \leq ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}},
\end{aligned}
$$

while,

$$
\begin{aligned}
&\left| \int w_{xx}\, (\partial^j v_{xx})\, (\partial^j v_{xxx}) \right| \\
&\quad = \; c \left| \int w_{xxx}\, (\partial^j v_{xx})^2 \right| \\
&\quad \leq \; c \left(\int w_{xxx}^2 \right)^{1/2} \left(\int (\partial^j v_{xx})^4 \right)^{1/2} \\
&\quad \leq \; c\, ||w||_{Z^0_t} \left(\int [\, (\partial^j v_{xx})^2 + (\partial^j v_{xxx})^2 + (\partial^j v_{xy})^2 \,] \right)^{1/2} \\
&\quad \leq \; c\, ||w||_{Z^{|j|}_t}\, ||v||^2_{Z^{|j|}_t}.
\end{aligned}
$$

For I_3,

$$
\begin{aligned}
&\left| \int \partial^j (w_x\, v_{xxx})\, (\partial^j v_{xxx}) \right| \\
&\quad = \; \left| \int [\, (\partial^j w_x)\, v_{xxx} + \, \ldots \, + w_x\, (\partial^j v_{xxx}) \,]\, (\partial^j v_{xxx}) \right| \\
&\quad \leq \; ||\partial^j w_x||_{L^\infty(\mathbb{R}^2)} \left(\int v_{xxx}^2 \right)^{1/2} \left(\int (\partial^j v_{xxx})^2 \right)^{1/2} \\
&\qquad + \, \ldots \, + ||w_x||_{L^\infty(\mathbb{R}^2)} \left(\int (\partial^j v_{xxx})^2 \right)^{1/2} \\
&\quad \leq \; c\, ||w||_{Z^{|j|}_t}\, ||v_{Z^0_t}\, ||v||_{Z^{|j|}_t} + \, \ldots \, + c\, ||w||_{Z^0_t}\, ||v||^2_{Z^{|j|}_t} \\
&\quad \leq \; c\, ||w||_{Z^{|j|}_t}\, ||v||^2_{Z^{|j|}_t}.
\end{aligned}
$$

Lastly, for I_4,

$$
\left| \int \partial^j (w\, v_{xxxx})\, (\partial^j v_{xxx}) \right| = \left| \int [\, (\partial^j w)\, v_{xxxx} + \, \ldots \, + w\, (\partial^j v_{xxxx}) \,]\, (\partial^j v_{xxx}) \right|.
$$

The first term is handled below. If $j = (0,\, 0)$, then

$$
\begin{aligned}
\left| \int (\partial^j w)\, v_{xxxx}\, (\partial^j v_{xxx}) \right| \; &= \; \left| \int w\, v_{xxxx}\, v_{xxx} \right| \\
&= \; c \left| \int w_x\, v_{xxx}^2 \right| \\
&\leq \; c\, ||w_x||_{L^\infty(\mathbb{R}^2)} \left(\int v_{xxx}^2 \right) \\
&\leq \; c\, ||w||_{Z^{|j|}_t}\, ||v||^2_{Z^0_t},
\end{aligned}
$$

while, if $|j| > 0$, then

$$
\begin{aligned}
\left| \int (\partial^j w)\, w_{xxxx}\, (\partial^j v_{xxx}) \right| \; &\leq \; ||\partial^j w||_{L^\infty(\mathbb{R}^2)} \left(\int v_{xxxx}^2 \right)^{1/2} \left(\int (\partial^j v_{xxx})^2 \right)^{1/2} \\
&\leq \; ||w||_{Z^{|j|}_t}\, ||v||^2_{Z^{|j|}_t}.
\end{aligned}
$$

The last term in I_4 is handled below

$$\begin{aligned}
\left|\int w\,(\partial^j v_{xxxx})\,(\partial^j v_{xxx})\right| &= c\left|\int w_x\,(\partial^j v_{xxx})^2\right| \\
&\le c\,||w_x||_{L^\infty(\mathbb{R}^2)}\left(\int(\partial^j v_{xxx})^2\right) \\
&\le c\,||w||_{Z_t^0}\,||v||^2_{Z_t^{|j|}} \\
&\le c\,||w||_{Z_t^{|j|}}\,||v||^2_{Z_t^{|j|}}.
\end{aligned}$$

Consequently, we conclude

$$\partial_t\int(\partial^j v_{xxx})^2 \le c\,||w||_{Z_t^{|j|}}\,||v||^2_{Z_t^{|j|}}.$$

Next we take two y derivatives of (3.8), multiply by $2\,(\partial^j v_{yy})$, and integrate over $\mathbb{R}^2$. Therefore, we have

$$\begin{aligned}
\partial_t\int(\partial^j v_{yy})^2 &\le c\left|\int\partial^j(w\,v_x)_{yy}\,(\partial^j v_{yy})\right| \\
&\le c\left|\int\partial^j(w_{yy}\,v_x+2\,w_y\,v_{xy}+w\,v_{xyy})\,(\partial^j v_{yy})\right| \\
&\le c\left|\int\partial^j(w_{yy}\,v_x)\,(\partial^j v_{yy})\right|+c\left|\int\partial^j(w_y\,v_{xy})\,(\partial^j v_{yy})\right| \\
&\quad\ +c\left|\int\partial^j(w\,v_{xyy})\,(\partial^j v_{yy})\right| \\
&\le I_5+I_6+I_7.
\end{aligned}$$

First, we look at I_5,

$$\begin{aligned}
&\left|\int\partial^j(w_{yy}\,v_x)\,(\partial^j v_{yy})\right| \\
&= \left|\int[\,(\partial^j w_{yy})\,v_x+\ \dots\ +w_{yy}\,(\partial^j v_x)\,]\,(\partial^j v_{yy})\right| \\
&\le c\,||v_x||_{L^\infty(\mathbb{R}^2)}\left(\int(\partial^j w_{yy})^2\right)^{1/2}\left(\int(\partial^j v_{yy})^2\right)^{1/2} \\
&\quad +\ \dots\ +c\,||\partial^j v_x||_{L^\infty(\mathbb{R}^2)}\left(\int w_{yy}^2\right)^{1/2}\left(\int(\partial^j v_{yy})^2\right)^{1/2} \\
&\le c\,||v||_{Z_t^0}\,||w||_{Z_t^{|j|}}\,||v||_{Z_t^{|j|}}+\ \dots\ +c\,||v||_{Z_t^{|j|}}\,||w||_{Z_t^0} \\
&\le c\,||w||_{Z_t^{|j|}}\,||v||^2_{Z_t^{|j|}}.
\end{aligned}$$

For I_6,

$$\begin{aligned}
&\left|\int \partial^j (w_y\, v_{xy})\, (\partial^j v_{yy})\right| \\
&\quad = \left|\int [\, (\partial^j w_y)\, v_{xy} + \,\ldots\, + w_y\, (\partial^j v_{xy})\,]\, (\partial^j v_{yy})\right| \\
&\quad \leq c\, ||w_y||_{L^\infty(\mathbb{R}^2)} \left(\int (\partial^j v_{xy})^2\right)^{1/2} \left(\int (\partial^j v_{yy})^2\right)^{1/2} \\
&\qquad + \,\ldots\, + c\, ||\partial^j w_y||_{L^\infty(\mathbb{R}^2)} \left(\int v_{xy}^2\right)^{1/2} \left(\int (\partial^j v_{yy})^2\right)^{1/2} \\
&\quad \leq c\, ||w||_{Z_t^{|j|}}\, ||v||_{Z_t^0}\, ||v||_{Z_t^{|j|}} + \,\ldots\, + c\, ||w||_{Z_t^0}\, ||v||_{Z_t^0} \\
&\quad \leq c\, ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}.
\end{aligned}$$

Lastly, for I_7,

$$\left|\int \partial^j (w\, v_{xyy})\, (\partial^j v_{yy})\right| = \left|\int [\, (\partial^j w)\, v_{xyy} + \,\ldots\, + w\, (\partial^j v_{xyy})\,]\, (\partial^j v_{yy})\right|.$$

We will look at the first and last of these terms below. The rest of these terms are handled similarly. For the first term, if $j = (0,\, 0)$, then we have

$$\begin{aligned}
\left|\int (\partial^j w)\, v_{xyy}\, (\partial^j v_{yy})\right| &= \left|\int w\, v_{xyy}\, v_{yy}\right| \\
&= c \left|\int w_x\, v_{yy}^2\right| \\
&\leq c\, ||w_x||_{L^\infty(\mathbb{R}^2)} \int v_{yy}^2 \\
&\leq c\, ||w||_{Z_t^0}\, ||v||^2_{Z_t^{|j|}},
\end{aligned}$$

while for $|j| > 0$,

$$\begin{aligned}
\left|\int (\partial^j w)\, v_{xyy}\, (\partial^j v_{yy})\right| &\leq ||\partial^j w||_{L^\infty(\mathbb{R}^2)} \left(\int v_{xyy}^2\right)^{1/2} \left(\int (\partial^j v_{yy})^2\right)^{1/2} \\
&\leq ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}.
\end{aligned}$$

The last term for I_7 is bounded as follows,

$$\begin{aligned}
\left|\int w\, (\partial^j v_{xyy})\, (\partial^j v_{yy})\right| &= c \left|\int w_x\, (\partial^j v_{yy})^2\right| \\
&\leq c\, ||w_x||_{L^\infty(\mathbb{R}^2)} \int (\partial^j v_{yy})^2 \\
&\leq c\, ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}.
\end{aligned}$$

Now apply one t derivative to (3.8), multiply by $2\,(\partial^j v_t)$ and integrate over $\mathbb{R}^2$. We arrive at the following inequality,

$$
\begin{aligned}
\partial_t \int (\partial^j v_t) &\le c \left| \int (\partial^j (w\, v_x)_t)\, (\partial^j v_t) \right| \\
&\le c \left| \int \partial^j (w_t\, v_x)\, (\partial^j v_t) \right| + c \left| \int \partial^j (w\, v_{xt})\, (\partial^j v_t)\, dx\, dt \right| \\
&= I_9 + I_{10}.
\end{aligned}
$$

For I_9, we have

$$
\begin{aligned}
&\left| \int \partial^j (w_t\, v_x)\, (\partial^j v_t) \right| \\
&\quad \le c \left| \int (\partial^j w_t)\, v_x\, (\partial^j v_t) \right| + \ldots + c \left| \int w_t\, (\partial^j v_x)\, (\partial^j v_t) \right| \\
&\quad \le c\, ||v_x||_{L^\infty(\mathbb{R}^2)} \left(\int (\partial^j w_t)^2 \right)^{1/2} \left(\int (\partial^j v_t)^2 \right)^{1/2} \\
&\qquad + \ldots + ||\partial^j v_x||_{L^\infty(\mathbb{R}^2)} \left(\int (w_t)^2 \right)^{1/2} \left(\int (\partial^j v_t)^2 \right)^{1/2} \\
&\quad \le c\, ||v||_{Z_t^0}\, ||w||_{Z_t^{|j|}}\, ||v||_{Z_t^{|j|}} + \ldots + c\, ||v||_{Z_t^{|j|}}\, ||w||_{Z_t^0}\, ||v||_{Z_t^{|j|}} \\
&\quad \le c\, ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}.
\end{aligned}
$$

Next we look at I_{10}. If $j = (0,\, 0)$, we have

$$
\begin{aligned}
\left| \int w\, v_{xt}\, v_t \right| &= \left| \int w_x\, v_t^2 \right| \\
&\le ||w_x||_{L^\infty(\mathbb{R}^2)} \int v_t^2 \\
&\le c\, ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}.
\end{aligned}
$$

If $j \ne (0,\, 0)$, we have

$$
\begin{aligned}
\left| \int \partial^j (w\, v_{xt})\, (\partial^j v_t) \right| &= c \left| \int (\partial^j w)\, v_{xt}\, (\partial^j v_t) \right| + \ldots \\
&\quad + c \left| \int w\, (\partial^j v_{xt})\, (\partial^j v_t) \right| \\
&= I_{10}(a) + \ldots + I_{10}(\widetilde{a}).
\end{aligned}
$$

Now for $I_{10}(a)$, we use the following estimate

$$
\begin{aligned}
\left| \int (\partial^j w)\, v_{xt}\, (\partial^j v_t) \right| &\le c\, ||\partial^j w||_{L^\infty(\mathbb{R}^2)} \left(\int v_{xt}^2 \right)^{1/2} \left(\int (\partial^j v_t)^2 \right)^{1/2} \\
&\le c\, ||w||_{Z_t^{|j|}}\, ||v||^2_{Z_t^{|j|}}.
\end{aligned}
$$

While for $I_{10}(\widetilde{a})$, we use the following estimate

$$\begin{aligned}
\left|\int w\,(\partial^j v_{xt})\,(\partial^j v_t)\right| &= c\left|\int w_x\,(\partial^j v_t)^2\right| \\
&\leq c\,||w_x||_{L^\infty(\mathbb{R}^2)}\int (\partial^j v_t)^2 \\
&\leq c\,||w||_{Z_t^{|j|}}\,||v||^2_{Z_t^{|j|}}.
\end{aligned}$$

Therefore, for $0 \leq \widetilde{t} \leq t$, we conclude that

$$\partial_t \int [\,(\partial^j v(\,\cdot\,,\,\cdot\,,\,\widetilde{t}))^2 + (\partial^j v_{xxx}(\,\cdot\,,\,\cdot\,,\,\widetilde{t}))^2 + (\partial^j v_{yy}(\,\cdot\,,\,\cdot\,,\,\widetilde{t}))^2 + (\partial^j v_t(\,\cdot\,,\,\cdot\,,\,\widetilde{t}))^2\,]$$
$$\leq c\,||w||_{Z_t^{|j|}}\,||v||^2_{Z_t^{|j|}}.$$

Integrating with respect to t, we obtain

$$||v||^2_{Z_t^{|j|}} \leq ||v(\,\cdot\,,\,\cdot\,,\,0))||^2_{H^{(|j|+3,\,|j|+2)}(\mathbb{R}^2)} + ||v(\,\cdot\,,\,\cdot\,,\,0))||^2_{H^{|j|}(\mathbb{R}^2)} + c\,t\,||w||_{Z_t^{|j|}}\,||v||^2_{Z_t^{|j|}},$$

as desired. $\square$

4 Uniqueness and Existence of a local solution

In this section, we will prove that for $\phi \in X^N(\mathbb{R}^2)$ there exists a unique solution of (2.1)-(2.2) in $L^\infty([0,\,T]\,:\,X^N(\mathbb{R}^2))$, where the time T depends only $||\phi||_{X^0(\mathbb{R}^2)}$. First we prove uniqueness of solutions.

Theorem 4.1 (Uniqueness). *Let $\phi \in X^0(\mathbb{R}^2)$ and $0 < T < +\infty$. Then there is at most one solution of* (2.1)-(2.2) *in $L^\infty([0,\,T]\,:\,X^0(\mathbb{R}^2))$ with initial data $u(x,\,y,\,0) = \phi(x,\,y)$.*

Proof. Assume that $u,\ v \in L^\infty([0,\,T]\,:\,X^0(\mathbb{R}^2))$ are two solutions of (2.1)-(2.2) with $u_t,\ v_t \in L^\infty([0,\,T]\,:\,L^2(\mathbb{R}^2))$, so all integrations below are justified and with the same initial data, in fact, with $(u-v)(x,\,y,\,0) = 0$. Then

$$(u-v)_t + (u-v)_{xxx} + (u-v)_x + (u\,u_x - v\,v_x) - \partial_x^{-1}(u-v)_{yy} = 0. \tag{4.1}$$

By (4.1),

$$(u-v)_t + (u-v)_{xxx} + (u-v)_x + (u-v)\,u_x + (u-v)_x\,v - \partial_x^{-1}(u-v)_{yy} = 0. \tag{4.2}$$

Multiplying (4.2) by $2\,(u-v)$ and integrating with respect to $(x,\,y)$ over $\mathbb{R}^2$,

$$\begin{aligned}
&2\int (u-v)\,(u-v)_t + 2\int (u-v)\,(u-v)_{xxx} + 2\int (u-v)\,(u-v)_x \\
&+ 2\int (u-v)^2\,u_x + 2\int (u-v)\,(u-v)_x\,v - 2\int (u-v)\,\partial_x^{-1}(u-v)_{yy} = 0.
\end{aligned} \tag{4.3}$$

Integrating by parts each term in (4.3) we obtain

$$\begin{aligned}\partial_t \int (u-v)^2 &= -2\int (u-v)^2\, u_x + \int (u-v)^2\, v_x \\ &\le c\left(\|u_x\|_{L^\infty(\mathbb{R}^2)} + \|v_x\|_{L^\infty(\mathbb{R}^2)}\right)\int (u-v)^2 \\ &\le c\left(\,\|u\|_{X^0(\mathbb{R}^2)} + \|v\|_{X^0(\mathbb{R}^2)}\,\right)\int (u-v)^2 \end{aligned} \tag{4.4}$$

Using Gronwall's inequality and the fact that $(u-v)$ vanishes at $t=0$, it follows that $u=v$. This proves the uniqueness of the solution. □

Now we consider existence of solutions to (2.1)-(2.2). Our plan is to show that for $\phi \in X^N(\mathbb{R}^2)$ there exists a solution $u \in L^\infty([0,T] : X^N(\mathbb{R}^2))$ for a time T depending only on $\|\phi\|_{X^0}$. In order to prove this we must first prove a preliminary result by introducing the following function space. Let

$$Y^N(\mathbb{R}^2) = \left\{ u :\; u,\, u_{xxx},\, u_{yy},\, \frac{\eta^2}{\xi}\,\widehat{u} \in H^N(\mathbb{R}^2) \right\} \tag{4.5}$$

with the accompanying norm

$$\|u\|^2_{Y^N(\mathbb{R}^2)} = \int_{\mathbb{R}^2}\left(u^2 + \sum_{|j|\le N} \left[\, (\partial^j u_{xxx})^2 + (\partial_x^{-1}\partial^j u_{yy})^2 + (\partial^j u_{yy})^2 \,\right]\right) dx\, dy \tag{4.6}$$

where $j=(\alpha_1,\, \alpha_2)$ and $|j| = \alpha_1+\alpha_2$. We will begin by showing that, for $\phi \in Y^N(\mathbb{R}^2)$, there exists a solution u of (2.1) such that $u \in L^\infty([0,T] : Y^N(\mathbb{R}^2))$ for a time T depending only on $\|\phi\|_{Y^0}$. Then we will prove a differential inequality of the form

$$\partial_t \left(\int u^2 + u_{xxx}^2 + (\partial_x^{-1} u_{yy})^2 \right) \le \left(\int u^2 + u_{xxx}^2 + (\partial_x^{-1} u_{yy})^2 \right)^{3/2},$$

to show that in fact the solution u obtained in Theorem 4.2 is in $L^\infty([0,T'];X^0(\mathbb{R}^2))$ for a time T' depending only on $\|\phi\|_{X^0(\mathbb{R}^2)}$. With these ideas in mind we state our existence theorem.

Theorem 4.2 (Existence). *Let $k_0 > 0$ and N be an integer ≥ 0. Then there exists a time $0 < T < +\infty$ depending only on k_0 such that for all $\phi \in Y^N(\mathbb{R}^2)$ with $\|\phi\|_{Y^0(\mathbb{R}^2)} \le k_0$ there exists a solution of* (2.1), *$u \in L^\infty\left([0,\, T] :\, Y^N(\mathbb{R}^2)\right)$ such that $u(x,\, y,\, 0) = \phi(x,\, y)$.*

The method of proof is as follows. As discussed in section 3, we begin by approximating (2.1) by the linear equation (3.1). We construct the mapping

$$\Pi : Z_T^N \to Z_T^N$$

where the initial condition is given by $u^{(n)}(x,\, y,\, 0) = \phi(x,\, y)$ and the first approximation is given by $u^{(0)}(x, y, t) = \phi(x, y)$. Subsequent approximations are given by $u^{(n)} = \Pi(u^{(n-1)})$ for $n \geq 1$. Equation (3.1) is a linear equation which by Lemma 3.1 can be solved at each iteration. We show that the sequence of solutions $\{u^{(n)}\}$ to our linear equation is bounded in $L^\infty([0, T]; Y^0(\mathbb{R}^2))$ for a time T depending only on $||\phi||_{Y^0}$. We then show that there is a subsequence of solutions to our approximate equations which converges to a solution $u \in L^\infty([0, T]; Y^0(\mathbb{R}^2))$ of (2.1). Lastly, we show that if $\phi \in Y^N(\mathbb{R}^2)$ for $N > 0$, then our solution $u \in L^\infty([0, T]; Y^N(\mathbb{R}^2))$ where the time T depends only on $||\phi||_{Y^0}$.

Proof. It suffices to prove this result for $\phi \in \bigcap_{N\geq 0} H^N(\mathbb{R}^2)$ and $\partial_x^{-1}\phi_{yy} \in \bigcap_{N\geq 0} H^N(\mathbb{R}^2)$. We can then use the same approximation procedure as before to prove the result for general initial data. Let $u^{(n)}$ be a solution of (4.5) with initial data $u^{(n)}(x, y, 0) = \phi(x, y)$ and where the first approximation is given by $u^{(0)}(x, y, t) = \phi(x, y)$. By Lemma 3.2, we know that

$$||u^{(n)}||^2_{Z^0_t} \leq ||u^{(n)}(\,\cdot\,,\,\cdot\,,\,0)||^2_{H^{(3,\,2)}(\mathbb{R}^2)} + ||u^{(n)}_t(\,\cdot\,,\,\cdot\,,\,0)||^2_{L^2(\mathbb{R}^2)} + c\,t\,||u^{(n-1)}||_{Z^0_t}\,||u^{(n)}||^2_{Z^0_t}. \quad (4.7)$$

Further, using the fact that $||\phi||_{Y^0} \leq k_0$, we have

$$\begin{aligned}
&||u^{(n)}(\,\cdot\,,\,\cdot\,,\,0)||^2_{H^{(3,\,2)}(\mathbb{R}^2)} + ||u^{(n)}_t(\,\cdot\,,\,\cdot\,,\,0)||^2_{L^2(\mathbb{R}^2)} \\
&\quad= \; ||u^{(n)}(\,\cdot\,,\,\cdot\,,\,0)||^2_{H^{(3,\,2)}(\mathbb{R}^2)} \\
&\qquad + \int [\,u^{(n)}_{xxx}(\,\cdot\,,\,\cdot\,,\,0) + u^{(n)}_x(\,\cdot\,,\,\cdot\,,\,0) - \partial_x^{-1}u^{(n)}_{yy}(\,\cdot\,,\,\cdot\,,\,0) + u^{(n-1)}(\,\cdot\,,\,\cdot\,,\,0)u^{(n)}_x(\,\cdot\,,\,\cdot\,,\,0)\,]^2 \\
&\quad\leq \; ||\phi||^2_{Y^0(\mathbb{R}^2)} + c\int [\,\phi^2_{xxx} + \phi^2_x + (\partial_x^{-1}\phi_{yy})^2 + (\phi\,\phi_x)^2\,] \\
&\quad\leq \; C\,||\phi||^2_{Y^0(\mathbb{R}^2)} \leq C\,k_0^2,
\end{aligned}$$

where C is independent of n. Define $c_0 = \left(\frac{C}{2}k_0^2 + 1\right)$. Let $T_0^{(n)}$ be the maximum time such that $||u^{(j)}||_{Z^0_t} \leq c_0$ for $0 \leq t \leq T_0^{(n)}$, $0 \leq j \leq n$. That is

$$T_0^{(n)} = \sup\{t \in [0,\, T_0^{(n)}] :\; ||u^{(j)}||_{Z^0_t} \leq c_0 \quad \text{for} \quad 0 \leq j \leq n\}.$$

Therefore,

$$\begin{aligned}
||u^{(n)}||^2_{Z^0_t} \;&\leq\; ||u^{(n)}(\,\cdot\,,\,\cdot\,,\,0)||^2_{H^{(3,\,2)}(\mathbb{R}^2)} + ||u^{(n)}_t(\,\cdot\,,\,\cdot\,,\,0)||^2_{L^2(\mathbb{R}^2)} + c\,t\,||u^{(n-1)}||_{Z^0_t}\,||u^{(n)}||^2_{Z^0_t} \\
&\leq\; C\,k_0^2 + c\,t\,c_0^3. \qquad (4.8)
\end{aligned}$$

Claim: $T_0^{(n)}$ does not approach 0.

On the contrary, assume that $T_0^{(n)} \to 0$. Since $||u^{(n)}(\,\cdot\,,\,\cdot\,,\,t)||_{Z^0_t}$ is continuous for $t \geq 0$, there exists $\tau \in [0,\, T]$ such that $c_0 = ||u^{(j)}(\,\cdot\,,\,\cdot\,,\,\tau)||_{Z^0_\tau}$ for $0 \leq \tau \leq T_0^{(n)}$, $0 \leq j \leq n$. Then, by (4.8) we have

$$c_0^2 \leq C\,k_0^2 + c\,T_0^{(n)}\,c_0^3. \qquad (4.9)$$

As $n \to \infty$, we have

$$\left(\frac{C}{2}\, k_0^2 + 1\right)^2 \leq C\, k_0^2 \quad \Longrightarrow \quad \frac{C^2}{4}\, k_0^4 + 1 \leq 0 \tag{4.10}$$

which is a contradiction. Consequently $T_0^{(n)} \not\to 0$. Choosing $T = T(c_0)$ sufficiently small, and T not depending on n, one concludes that

$$||u^{(n)}||^2_{Z_t^0} \leq c \quad \text{for} \quad 0 \leq t \leq T. \tag{4.11}$$

This show that $T_0^{(n)} \geq T$. Hence from (4.11) we see that there exists a bounded sequence of solutions $u^{(n)} \in Z_T^0$ and therefore a subsequence $u^{(n_j)} \equiv u^{(n)}$ such that

$$u^{(n)} \overset{*}{\rightharpoonup} u \quad \text{weakly in} \quad L^\infty([0,\, T] :\, H^{(3,\,2)}(\mathbb{R}^2))$$

$$u_t^{(n)} \overset{*}{\rightharpoonup} u_t \quad \text{weakly in} \quad L^\infty([0,\, T] :\, L^2(\mathbb{R}^2)).$$

Therefore, by Lions-Aubin's compactness theorem there is a subsequence $u^{(n_j)} \equiv u^{(n)}$ such that $u^{(n)} \to u$ strongly on $L^\infty([0,\, T] :\, H^1_{loc}(\mathbb{R}^2))$. Now it remains to show that each term in (3.1) converges to its correct limit. First, $u_{xxx}^{(n)} \overset{*}{\rightharpoonup} u_{xxx}$ weakly on $L^\infty([0,\, T] :\, L^2(\mathbb{R}^2))$. Similarly $u_t^{(n)} \to u_t$ and $u_x^{(n)} \to u_x$ weak* in $L^\infty([0,\, T] :\, L^2(\mathbb{R}^2))$. Now we will show that the nonlinear term converges to its correct limit. First, $u^{(n-1)} \to u$ strongly in $L^\infty([0,\, T] :\, H^1_{loc}(\mathbb{R}^2))$. Moreover, $u_x^{(n-1)} \overset{*}{\rightharpoonup}$ weakly in $L^\infty([0,\, T] :\, L^2(\mathbb{R}^2))$. Therefore,

$$u^{(n-1)}\, u_x^{(n)} \overset{*}{\rightharpoonup} u\, u_x \quad \text{weakly in} \quad L^\infty([0,\, T] :\, L^2(\mathbb{R}^2)).$$

Consequently,

$$\begin{aligned} \partial_x^{-1} u_{yy}^{(n)} &= u_t^{(n)} + u_{xxx}^{(n)} + u_x^{(n)} + u^{(n)}\, u_x^{(n)} \\ &\overset{*}{\rightharpoonup} u_t + u_{xxx} + u_x + u\, u_x \quad \text{weakly in} \quad L^\infty([0,\, T] :\, L^2(\mathbb{R}^2)). \end{aligned} \tag{4.12}$$

But, also note that

$$u_{yy}^{(n)} \overset{*}{\rightharpoonup} u_{yy} \quad \text{weakly in} \quad L^\infty([0,\, T] :\, L^2(\mathbb{R}^2)).$$

Therefore

$$\partial_x^{-1} u_{yy}^{(n)} \overset{*}{\rightharpoonup} \partial_x^{-1} u_{yy} \quad \text{weakly in} \quad L^\infty([0,\, T] :\, L^2(\mathbb{R}^2))$$

and consequently u is a solution to (2.1). Now, we prove that there exists a solution to (2.1) with $u \in L^\infty([0,\, T] :\, Y^N(\mathbb{R}^2))$ for the time T chosen above. We already know that there is a solution $u \in L^\infty([0,\, T] :\, Y^0(\mathbb{R}^2))$. Therefore, it suffices to show that the approximating sequence $u^{(n)}$ is bounded in Z_T^N and thus, by the convergence arguments above, our solution u is in $L^\infty([0,\, T] :\, Y^N(\mathbb{R}^2))$. Again, by Lemma 3.1, we know our linearized equation can

be solved in any interval of time in which the coefficients are defined. Therefore, for each iterate, $||u^{(n)}||_{Z_t^N}$ is continuous in $t \in [0, T]$. By Lemma 3.2, it follows that

$$\begin{aligned} ||u^{(n)}||^2_{Z_t^N} \;\; &\le \;\; ||u^{(n)}(\,\cdot\,,\,\cdot\,,0)||^2_{H^{(N+3,\,N+2)}(\mathbb{R}^2)} \\ &\quad + ||u_t^{(n)}(\,\cdot\,,\,\cdot\,,0)||^2_{H^N(\mathbb{R}^2)} + c\,t\,||u^{(n-1)}||_{Z_t^N}\,||u^{(n)}||^2_{Z_t^N}. \end{aligned} \tag{4.13}$$

On the other hand, as before and using $||\phi||_{Y^N} \le k_N$ we obtain

$$||u^{(n)}(\,\cdot\,,\,\cdot\,,0)||^2_{H^{(N+3,\,N+2)}(\mathbb{R}^2)} + ||u_t^{(n)}(\,\cdot\,,\,\cdot\,,0)||^2_{H^N(\mathbb{R}^2)} \;\; \le C\,k_N^2,$$

where k_N is independent of n. Define $c_N = \left(\frac{C}{2}\,k_N^2 + 1\right)$. Let $T_N^{(n)}$ be the largest time that $||u^{(j)}||_{Z_t^N} \le c_N$ for $0 \le t \le T_N^{(n)}$, $0 \le j \le n$. That is,

$$T_N^{(n)} = \sup\{t \in [0,\,T_N^{(n)}] :\; ||u^{(j)}||_{Z_t^N} \le c_N \quad \text{for} \quad 0 \le j \le n\}.$$

Therefore, for $0 \le t \le T_N^{(n)}$,

$$||u^{(n)}||^2_{Z_t^N} \le C\,k_N^2 + c\,t\,c_N^3. \tag{4.14}$$

Claim: $T_N^{(n)}$ does not approach 0.

On the contrary, assume that $T_N^{(n)} \to 0$. Since $||u^{(n)}(\,\cdot\,,\,\cdot\,,t)||_{Z_t^N}$ is continuous for $t \ge 0$, there exists $\tau \in [0, T]$ such that $c_N = ||u^{(j)}(\,\cdot\,,\,\cdot\,,\tau)||_{Z_\tau^N}$ for $0 \le \tau \le T_N^{(n)}$, $0 \le j \le n$. Then, by (4.14) we have

$$c_N^2 \le C\,k_N^2 + c\,T_N^{(n)}\,c_N^3. \tag{4.15}$$

As $n \to \infty$, we have

$$\left(\frac{C}{2}\,k_N^2 + 1\right)^2 \le C\,k_N^2 \quad \Longrightarrow \quad \frac{C^2}{4}\,k_N^4 + 1 \le 0 \tag{4.16}$$

which is a contradiction. Consequently $T_N^{(n)} \not\to 0$. Choosing T_N sufficiently small, and T_N not depending on n, one concludes that

$$||u^{(n)}||^2_{Z_t^N} \le c \quad \text{for} \quad 0 \le t \le T_N. \tag{4.17}$$

This show that $T_N^{(n)} \ge T_N$. Now, let

$$T_N^* = \sup\{t \in [0,\,T_N^*] :\; u \in Z_t^N\}.$$

We claim that $T_N^* \ge T$ and therefore, a time of existence can be chosen depending only on $||\phi||_{Y^0}$. By Lemma 3.1 the linear equation (3.1) can be solved in any interval of time in which the coefficients are defined, and thus $T_N^* \ge T$. $\square$

Now we want to improve our existence theorem. In particular, we want to show that the solution $u \in L^\infty([0,\, T] \,:\, Y^N(\mathbb{R}^2))$ found in Theorem 4.2 is in $L^\infty([0,\, T'] \,:\, X^N(\mathbb{R}^2))$ for a time T' depending only on $||\phi||_{X^0(\mathbb{R}^2)}$. In order to do so, we first prove a differential inequality.

Lemma 4.3. *Let u be the solution to our main equation in $L^\infty([0,\, T] \,:\, Y^N(\mathbb{R}^2))$. Then for any $0 \le t \le T$, we have*

$$\begin{aligned} &\partial_t \int_{\mathbb{R}^2} \left(u^2 + \sum_{|j| \le N} [\, (\partial^j u_{xxx})^2 + (\partial^j (\partial_x^{-1} u_{yy}))^2 \,] \right) dx\, dy \\ &\le \; c \left[\int_{\mathbb{R}^2} \left(u^2 + \sum_{|j| \le N} [\, (\partial^j u_{xxx})^2 + (\partial^j (\partial_x^{-1} u_{yy}))^2 \,] \right) dx\, dy \right]^{3/2} . \qquad (4.18) \end{aligned}$$

Proof. We use a priori estimates on smooth solutions u. Multiplying (2.1) by u and integrating over $\mathbb{R}^2$, it is straightforward to see that the $L^2(\mathbb{R}^2)$-norm is conserved. Therefore, we only need to show that

$$\begin{aligned} &\partial_t \int \sum_{|j| \le N} [\, (\partial^j u_{xxx})^2 + (\partial^j (\partial_x^{-1} u_{yy}))^2 \,] \\ &\le \; c \left[\int \left(u^2 + \sum_{|j| \le N} [\, (\partial^j u_{xxx})^2 + (\partial^j (\partial_x^{-1} u_{yy}))^2 \,] \right) \right]^{3/2} . \qquad (4.19) \end{aligned}$$

We consider the case $j = (0,\, 0)$. The case $j \neq (0,\, 0)$ is handled in a similar way.

Applying ∂_x^3 to (2.1) we obtain

$$u_{xxxt} + u_{xxxxxx} + u_{xxxx} + (u\, u_x)_{xxx} - u_{xxyy} = 0. \qquad (4.20)$$

Multiplying (4.20) by $2\, u_{xxx}$ and integrating over $\mathbb{R}^2$ we obtain

$$\begin{aligned} &2 \int u_{xxx}\, u_{xxxt} + 2 \int u_{xxx}\, u_{xxxxxx} + 2 \int u_{xxx}\, u_{xxxx} \\ &+ 2 \int u_{xxx}\, (u\, u_x)_{xxx} - 2 \int u_{xxx}\, u_{xxyy} = 0. \qquad (4.21) \end{aligned}$$

Using in (4.21) straightforward integration by parts, we obtain

$$\begin{aligned} \partial_t \int u_{xxx}^2 &= -\, 2 \int u_{xxx}\, (u\, u_x)_{xxx} \\ &= \; -\, 2 \int [\, 3\, u_{xx}^2 + 4\, u_x\, u_{xxx} + u\, u_{xxxx} \,]\, u_{xxx} \\ &= \; -\, 7 \int u_x\, u_{xxx}^2 \le 7\, ||u_x||_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} u_{xxx}^2 \\ &\le \; c \left(\int [\, u_x^2 + u_{xxx}^2 + u_{xy}^2 \,] \right)^{1/2} \int_{\mathbb{R}^2} u_{xxx}^2 \\ &\le \; c \left(\int [\, u^2 + u_{xxx}^2 + (\partial_x^{-1} u_{yy}^2)^2 \,] \right)^{3/2} . \end{aligned}$$

In a similar way, but now apply $\partial_x^{-1}\partial_y^2$ to (2.1) instead of ∂_x^3 and multiply by $2\,\partial_x^{-1}u_{yy}$ instead of $2\,u_{xxx}$ we get

$$
\begin{aligned}
&\partial_t \int (\partial_x^{-1}u_{yy})^2 \\
&\leq \; c\left|\int (u^2)_{yy}\,(\partial_x^{-1}u_{yy})\right| = c\left|\int [\,u_{yy}\,u + u_y^2\,]\,(\partial_x^{-1}u_{yy})\right| \\
&\leq \; c\left|\int u_x\,(\partial_x^{-1}u_{yy})^2\right| + \left|\int u_y^2\,(\partial_x^{-1}u_{yy})\right| \\
&\leq \; c||u_x||_{L^\infty(\mathbb{R}^2)}\int (\partial_x^{-1}u_{yy})^2 + \left(\int u_y^4\right)^{1/2}\left(\int (\partial_x^{-1}u_{yy})^2\right)^{1/2} \\
&\leq \; c||u_x||_{L^\infty(\mathbb{R}^2)}\int (\partial_x^{-1}u_{yy})^2 + \left(\int [\,u_y^2 + u_{xy}^2 + (\partial_x^{-1}u_{yy})^2\,]\right)\left(\int (\partial_x^{-1}u_{yy})^2\right)^{1/2} \\
&\leq \; c\left(\int [\,u^2 + u_{xxx}^2 + (\partial_x^{-1}u_{yy})^2\,]\right)^{3/2}.
\end{aligned}
$$

The lemma follows. □

Corollary 4.4. *Let u be the solution to* (2.1) *with initial data $\phi \in Y^N(\mathbb{R}^2)$. Denote by $0 < T < +\infty$ the life span of this solution in $Y^N(\mathbb{R}^2)$. Then there exists $0 < T' \leq T$, depending only on the norm of $\phi \in X^0(\mathbb{R}^2)$ such that $u \in L^\infty([0,\,T']:\,X^N(\mathbb{R}^2))$.*

Proof. Let

$$h(t) = \int_{\mathbb{R}^2}\left(u^2 + \sum_{|j|\leq N}[\,(\partial^j u_{xxx})^2 + (\partial^j(\partial_x^{-1}u_{yy}))^2\,]\right)dx\,dy \equiv ||u||^2_{X^N}.$$

Using (4.18) we have $h'(t) \leq c\,[\,h(t)\,]^{3/2}$. Integrating this inequality with respect to t, we obtain that $h(t)^{1/2} \leq c/(h(0)^{-1/2} - t)$ and therefore, we get a lower bound on the time of existence of $h(t)$ depending only on $h(0)$. □

Corollary 4.5. *Let $\phi \in X^N(\mathbb{R}^2)$ for some $N \geq 0$ and let $\phi^{(n)}$ be a sequence converging to ϕ in $X^N(\mathbb{R}^2)$. Let u and $u^{(n)}$ be the corresponding unique solutions, given by Theorems 4.1 and 4.2 and Corollary 4.4, in $L^\infty([0,\,T]:\,X^N(\mathbb{R}^2))$ for a time T depending only on $\sup_n||\phi^{(n)}||_{X^0(\mathbb{R}^2)}$. Then*

$$u^{(n)} \overset{*}{\rightharpoonup} u \quad \text{weakly in} \quad L^\infty([0,\,T]:\,X^N(\mathbb{R}^2)). \tag{4.22}$$

Proof. By assumption $u \in L^\infty([0,\,T]:\,X^N(\mathbb{R}^2))$, then there exists a weak* convergent subsequence, still denoted $u^{(n)}$ such that

$$u^{(n)} \overset{*}{\rightharpoonup} u \quad \text{weakly in} \quad L^\infty([0,\,T]:\,X^N(\mathbb{R}^2)) \hookrightarrow L^\infty([0,\,T]:\,H^1(\mathbb{R}^2)).$$

Moreover, by equation (2.1), $u^{(n)} \in L^\infty([0,\,T]:\,X^N(\mathbb{R}^2))$ implies $u_t^{(n)} \in L^\infty([0,\,T]:\,L^2(\mathbb{R}^2))$. By The Lions-Aubin compactness theorem,

$$u^{(n)} \to u \quad \text{strongly in} \quad L^\infty([0,\,T]:\,H^{1/2}_{loc}(\mathbb{R}^2)).$$

Now we just need to show that each term in (2.1) converges to its correct limit, and $u_t^{(n)} \to u_t$ for $u \in L^\infty([0,\, T] : X^N(\mathbb{R}^2))$.

The only thing we need to show is that the nonlinear term converges to its correct limit, namely that $u^{(n)}\, u_x^{(n)} \to u\, u_x$. We know that $u_x^{(n)} \overset{*}{\rightharpoonup} u_x$ weakly in $L^\infty([0,\, T] : H^1(\mathbb{R}^2))$ and $u^{(n)} \to u$ strongly in $L^\infty([0,\, T] : H^{1/2}_{loc}(\mathbb{R}^2))$. Therefore, their product converges in $L^2([0,\, T] : L^1_{loc}(\mathbb{R}^2))$. Clearly, the linear terms also converge in $L^2([0,\, T] : L^1_{loc}(\mathbb{R}^2))$ and therefore, we conclude that $u_t^{(n)} \to u_t$ in $L^2([0,\, T] : L^1_{loc}(\mathbb{R}^2))$. The proof follows. $\square$

5 Estimate of error terms

In this section we prove the main estimates used in our gain of regularity theorem.

Theorem 5.1 *Let $L \geq 2$. For u a solution of (2.1), sufficiently smooth and with sufficient decay at infinity,*

$$\sup_{0\leq t\leq T} \int f_\alpha(\partial^\alpha u)^2 + \int_0^T \int g_\alpha(\partial^\alpha u_x)^2 \leq C \tag{5.1}$$

for $L+1 \leq |\alpha| \leq 2L-1$, $2L-|\alpha|-\alpha_2 \geq 1$, where $f_\alpha \in W_{\sigma,2L-|\alpha|-\alpha_2,|\alpha|-L}$, $g_\alpha \in W_{\sigma,2L-|\alpha|-\alpha_2-1,|\alpha|-L}$ and C depends only on $||u||_{X^1}$ and

$$\sup_{0\leq t\leq T} \int f_\gamma(\partial^\gamma u)^2 \tag{5.2}$$

$$\int_0^T \int g_\gamma(\partial^\gamma u_x)^2 \tag{5.3}$$

where $\gamma = (\gamma_1,\gamma_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $|\gamma| \leq |\alpha|-1$, $f_\gamma \in W_{\sigma,2L-|\gamma|-\gamma_2,|\gamma|-L}, g_\gamma \in W_{\sigma,2L-|\gamma|-\gamma_2-1,|\gamma|-L}$ for $|\gamma| \geq L$, $2L-|\gamma|-\gamma_2 \geq 1$ and $f_\gamma \in W_{0,\gamma_1,0}, g_\gamma \in W_{\sigma,\gamma_1-1,0}$ for $0 \leq |\gamma| \leq L$.

The idea of the proof is the following. For a given α satisfying the hypotheses above, we choose a weight function $f_\alpha \approx t^{|\alpha|-L}x^{2L-|\alpha|-\alpha_2}$ for $x > 1$ and $f_\alpha \approx t^{|\alpha|-L}e^{\sigma x}$ for $x < -1$. Then with this choice of weight function, we apply the operator ∂^α to (2.1), multiply by $f_\alpha\partial^\alpha u$ and integrate over $\mathbb{R}^2$ to obtain the main equality stated in (2.3). In this theorem, we bound the last three terms on the left-hand side of (2.3) by terms of the form (5.2) and (5.3).

Proof. For each α we apply the operator ∂^α to (2.1), multiply our differentiated equation by $2f_\alpha(\partial^\alpha u)$ where we take

$$f_\alpha(x,t) = \int_{-\infty}^x g_\alpha(z,t)\,dz \quad \text{for } g_\alpha \in W_{\sigma,2L-|\alpha|-\alpha_2-1,|\alpha|-L}, \tag{5.4}$$

and integrate over $\mathbb{R}^2 \times [0,t]$ for $0 \leq t \leq T$. As stated in Lemma 2.1, we arrive at our main

equality

$$\partial_t \int f_\alpha(\partial^\alpha u)^2 + 3\int (f_\alpha)_x(\partial^\alpha u_x)^2 - \int [(f_\alpha)_t + (f_\alpha)_{xxx} + (f_\alpha)_x](\partial^\alpha u)^2$$
$$- \int (f_\alpha)_x(\partial^\alpha \partial_x^{-1} u_y)^2 + 2\int f_\alpha(\partial^\alpha u)\partial^\alpha(uu_x) = 0.$$

Using (1.5) and $f(\cdot,0)=0$ we get the following identity after integrating with respect to t,

$$\begin{aligned}
&\int f_\alpha(\partial^\alpha u)^2 + 3\int_0^T \int (f_\alpha)_x(\partial^\alpha u_x)^2 \\
&\le \int_0^T \int (f_\alpha)_x(\partial^\alpha \partial_x^{-1} u_y)^2 + C\int_0^T \int f_\alpha(\partial^\alpha u)^2 + 2\left|\int_0^T \int f_\alpha(\partial^\alpha u)\partial^\alpha(uu_x)\right|.
\end{aligned} \tag{5.5}$$

We notice that the first term on the right-hand side of (5.5) can be written as

$$\int_0^T \int (f_\alpha)_x(\partial^\alpha \partial_x^{-1} u_y)^2 = C\int_0^T \int t g_\gamma(\partial^\gamma u_x)^2 \tag{5.6}$$

for some $g_\gamma \in W_{\sigma,2L-|\gamma|-\gamma_2-1,|\gamma|-L}$ where $\gamma = (\alpha_1 - 2, \alpha_2 + 1)$. Further, we notice that $2L - |\gamma| - \gamma_2 \geq 1$ and $\alpha_1 \geq 2$ since $2L - |\alpha| - \alpha_2 \geq 1$ and $L + 1 \leq |\alpha|$. Therefore, (5.6) is of the form specified by (5.3). Therefore,

$$\begin{aligned}
&\int f_\alpha(\partial^\alpha u)^2 + 3\int_0^T \int (f_\alpha)_x(\partial^\alpha u_x)^2 \\
&\qquad \le C + C\int_0^T \int f_\alpha(\partial^\alpha u)^2 + 2\left|\int_0^T \int f_\alpha(\partial^\alpha u)\partial^\alpha(uu_x)\right|
\end{aligned} \tag{5.7}$$

where C depends only on terms of the form (5.3). We now need to estimate the term

$$\left|\int_0^T \int f_\alpha(\partial^\alpha u)\partial^\alpha(uu_x)\right|.$$

Each term is of the form

$$\left|\int_0^T \int f_\alpha(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\right|.$$

where $r_1 + s_1 = \alpha_1$, $r_2 + s_2 = \alpha_2$. Below we consider all terms of level $|\alpha|$.

The case $|s| = |\alpha|$. In this case, $|r| = 0$, and we have

$$\begin{aligned}
\left|\int_0^T \int f_\alpha(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\right| &= \left|\int_0^T \int f_\alpha(\partial^\alpha u)u(\partial^\alpha u_x)\right| \\
&= C\left|\int_0^T \int (f_\alpha u)_x(\partial^\alpha u)^2\right| \\
&\le C||u||_{X^0}\int_0^T \int f_\alpha(\partial^\alpha u)^2.
\end{aligned}$$

The case $|s| = |\alpha| - 1$**.** In this case, $|r| = 1$ giving us the following two subcases:

(a) **The subcase** $r = (1, 0)$**.** In this case, we have

$$\begin{aligned}\left|\int_0^T \int f_\alpha(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\right| &= \left|\int_0^T \int f_\alpha(\partial^\alpha u) u_x (\partial^\alpha u)\right| \\ &\leq C||u||_{X^0} \int_0^T \int f_\alpha(\partial^\alpha u)^2.\end{aligned}$$

(b) **The subcase** $r = (0, 1)$**.** We note that this case will only occur if $\alpha_2 \geq 1$. In this case, we have

$$\begin{aligned}\left|\int_0^T \int f_\alpha(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\right| &= \left|\int_0^T \int f_\alpha(\partial^\alpha u) u_y (\partial_x^{\alpha_1}\partial_y^{\alpha_2 - 1} u_x)\right| \\ &\leq C||u||_{X^1} \left(\int_0^T \int f_\alpha(\partial^\alpha u)^2\right)^{1/2} \left(\int_0^T \int f_\alpha(\partial_x^{\alpha_1+1}\partial_y^{\alpha_2 - 1} u)^2\right)^{1/2}\end{aligned}$$

Since $f_\alpha \approx x^{2L - |\alpha| - \alpha_2}$ as $x \to \infty$, it is clear that $f_\alpha \leq C f_{\alpha_1+1, \alpha_2 - 1}$

The case $|s| = |\alpha| - 2$**.** We have three subcases to consider.

(a) **The subcase** $r = (2, 0)$**.** In this case, we have

$$\begin{aligned}\left|\int_0^T \int f_\alpha(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\right| &= \left|\int_0^T \int f_\alpha(\partial^\alpha u) u_{xx} (\partial_x^{\alpha_1 - 2}\partial_y^{\alpha_2} u_x)\right| \\ &\leq C||u_{xx}||_{L^\infty} \left(\int_0^T \int f_\alpha(\partial^\alpha u)^2\right)^{1/2} \left(\int_0^T \int f_\alpha(\partial_x^{\alpha_1 - 1}\partial_y^{\alpha_2} u)^2\right)^{1/2} \\ &\leq C||u||_{X^1} \left(\int_0^T \int f_\alpha(\partial^\alpha u)^2\right)^{1/2} \left(\int_0^T \int f_\alpha(\partial_x^{\alpha_1 - 1}\partial_y^{\alpha_2} u)^2\right)^{1/2}.\end{aligned}$$

The last term on the right-hand side above is of order $|\alpha| - 1 \geq L$. The weight function $f_\alpha \in W_{\sigma, 2L - |\alpha| - \alpha_2, |\alpha| - L}$. Since $2L - |\alpha| - \alpha_2 < 2L - (\alpha_1 - 1 + \alpha_2) - \alpha_2$, we see that this term is bounded by a term of the form (5.2).

(b) **The subcase** $r = (1, 1)$**.** In this case, we have

$$\begin{aligned}\left|\int_0^T \int f_\alpha(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\right| &= \left|\int_0^T \int f_\alpha(\partial^\alpha u) u_{xy} (\partial_x^{\alpha_1 - 1}\partial_y^{\alpha_2 - 1} u_x)\right| \\ &\leq C||u_{xy}||_{L^\infty} \left(\int_0^T \int f_\alpha(\partial^\alpha u)^2\right)^{1/2} \left(\int_0^T \int f_\alpha(\partial_x^{\alpha_1}\partial_y^{\alpha_2 - 1} u)^2\right)^{1/2} \\ &\leq C||u||_{X^1} \left(\int_0^T \int f_\alpha(\partial^\alpha u)^2\right)^{1/2} \left(\int_0^T \int f_\alpha(\partial_x^{\alpha_1}\partial_y^{\alpha_2 - 1} u)^2\right)^{1/2}\end{aligned}$$

Using the fact that $f_\alpha \in W_{\sigma, 2L - |\alpha| - \alpha_2, |\alpha| - L}$ and $2L - |\alpha| - \alpha_2 < 2L - (\alpha_1 + \alpha_2 - 1) - (\alpha_2 - 1)$, we conclude that the last term is bounded by a term of the form (5.2).

(c) **The subcase** $r = (0, 2)$**.** In this case, we have

$$\left| \int_0^T \int f_\alpha (\partial^\alpha u)(\partial^r u)(\partial^s u_x) \right| = \left| \int_0^T \int f_\alpha (\partial^\alpha u) u_{yy} (\partial_x^{\alpha_1} \partial_y^{\alpha_2 - 2} u_x) \right|$$
$$\leq \left(\int_0^T \int f_\alpha (\partial^\alpha u)^2 \right)^{1/2} \left(\int_0^T \int u_{yy}^4 \right)^{1/4}$$
$$\times \left(\int_0^T \int (f_\alpha^{1/2} \partial_x^{\alpha_1 + 1} \partial_y^{\alpha_2 - 2} u)^4 \right)^{1/4}.$$

Now

$$\left(\int_0^T \int u_{yy}^4 \right)^{1/4} \leq C \left(\int_0^T \int [u_{yy}^2 + u_{xyy}^2 + (\partial_x^{-1} u_{yyy})^2] \right)^{1/2} \leq C ||u||_{X^1}.$$

Further,

$$\left(\int_0^T \int (f_\alpha^{1/2} \partial_x^{\alpha_1 + 1} \partial_y^{\alpha_2 - 2} u)^4 \right)^{1/4}$$
$$\leq C \left(\int_0^T \int f_\alpha (\partial_x^{\alpha_1 + 1} \partial_y^{\alpha_2 - 2} u)^2 + f_\alpha (\partial_x^{\alpha_1 + 2} \partial_y^{\alpha_2 - 2} u)^2 + f_\alpha (\partial_x^{\alpha_1} \partial_y^{\alpha_2 - 1} u)^2 \right)^{1/2}$$

Now the first and third terms in the integrand are of order $|\alpha| - 1$ and are clearly bounded by terms of the form (5.2). The second term in the integrand is of order $|\alpha|$. It will be bounded using Gronwall's inequality (and using the fact that the order of the y derivative is less than α_2 and therefore this terms can handle an even greater power of x.)

The case $|s| = |\alpha| - 3$**.** In this case $|r| = 3$. First, we consider the case in which $L \geq 4$. Since $|\alpha| \geq L + 1$, we note that $|s| + 2 = |\alpha| - 1 \geq L$. Using this fact, we bound as follows.

$$\left| \int_0^T \int f_\alpha (\partial^\alpha u)(\partial^r u)(\partial^s u_x) \right| \leq \left(\int_0^T \int f_\alpha (\partial^\alpha u)^2 \right)^{1/2} \left(\int_0^T \int (\partial^r u)^4 \right)^{1/4}$$
$$\times \left(\int_0^T \int (f_\alpha^{1/2} \partial^s u_x)^4 \right)^{1/4}$$

Now

$$\left(\int_0^T \int (\partial^r u)^4 \right)^{1/4} \leq C \left(\int_0^T \int [(\partial^r u)^2 + (\partial^r u_x)^2 + (\partial^r u_y)^2] \right)^{1/2}.$$

Since $|r| = 3$, each of these terms is at most of order $4 \leq L \leq |\alpha| - 1$, and, therefore, bounded by terms of the form (5.2). Similarly,

$$\left(\int_0^T \int (f_\alpha^{1/2} \partial^s u_x)^4 \right)^{1/4} \leq C \left(\int_0^T \int f_\alpha [(\partial^s u_x)^2 + (\partial^s u_{xx})^2 + (\partial^s u_{xy})^2] \right)^{1/2} \tag{5.8}$$

We notice that the last two terms on the right-hand side of (5.8) are of order $|s| + 2 = |\alpha| - 1 \geq L$. In order to verify that we have the correct power of x, we note that

$$2L - |\alpha| - \alpha_2 \leq 2L - (|s| + 2) - (s_2 + 1)$$

since $|s| = |\alpha| - 3$. Therefore, we can conclude that each of those terms is bounded by a term of the form (5.2). Finally, we look at the first term on the right-hand side of (5.8). If $|s| + 1 \geq L$, then this term is bounded by (5.2) as the other two terms. If $|s| + 1 < L$, then using the fact that $|s| + 2 \geq L$, we conclude that $|s| = L - 2$, and, therefore,

$$\begin{aligned} 2L - |\alpha| - \alpha_2 &\leq 2L - (|s| + 2) - (s_2 + 1) \\ &\leq s_1 + 1 \end{aligned}$$

Therefore, we conclude that the first term above is bounded by a term of the form (5.2) of order $|s| + 1 < L$.

We now look at the cases when $L = 2$ or $L = 3$. In either case, if $|\alpha| \geq 5$, then we can handle as above. We first consider the case when $|\alpha| = 4$ ($L = 2$ or $L = 3$). In this case, using the fact that $|r| = 3$ and $|s| = 1$, we have

$$\begin{aligned} \left| \int_0^T \int f_\alpha (\partial^\alpha u)(\partial^r u)(\partial^s u_x) \right| &\leq C ||\partial^s u_x||_{L^\infty} \left(\int_0^T \int f_\alpha (\partial^\alpha u)^2 \right)^{1/2} \left(\int_0^T \int f_\alpha (\partial^r u)^2 \right)^{1/2} \\ &\leq C ||u||_{X^1} \left(\int_0^T \int f_\alpha (\partial^\alpha u)^2 \right)^{1/2} \left(\int_0^T \int f_\alpha (\partial^r u)^2 \right)^{1/2}. \end{aligned}$$

Since $|r| = 3 = |\alpha| - 1$ and $2L - |\alpha| - \alpha_2 \leq 2L - |r| - r_2$, we see that the last term above is bounded by a term of the form (5.2).

Last, we consider $|\alpha| = 3$. In this case, we must have $L = 2$, $|r| = 3$ and $|s| = 0$. Therefore, $r = \alpha$. We bound as follows:

$$\begin{aligned} \left| \int_0^T \int f_\alpha (\partial^\alpha u)(\partial^r u)(\partial^s u_x) \right| &\leq C ||u_x||_{L^\infty} \int_0^T \int f_\alpha (\partial^\alpha u)^2 \\ &\leq C ||u||_{X^1} \int_0^T \int f_\alpha (\partial^\alpha u)^2. \end{aligned}$$

The case $|s| \leq |\alpha| - 4$. We consider the set $A = \{x : x > 1\}$. The set $A_{-1} = \{x < -1\}$ can be handled similarly. We have

$$\begin{aligned} &\left| \int_0^T \int_A f_\alpha (\partial^\alpha u)(\partial^r u)(\partial^s u_x) \right| \\ &\leq C T^M ||t^{\nu_s} \partial^s u_x||_{L^\infty(A)} \left(\int_0^T \int_A f_\alpha (\partial^\alpha u)^2 \right)^{1/2} \left(\int_0^T \int_A t^{\nu_r} x^{2L - |\alpha| - \alpha_2} (\partial^r u)^2 \right)^{1/2} \end{aligned}$$

where $\nu_s = \frac{(|s|+3-L)^+}{2}$ and $\nu_r = \frac{(|r|-L)^+}{2}$. First, we must verify that $M \geq 0$. We see that

$$\begin{aligned} M &= \frac{|\alpha| - L}{2} - \nu_s - \nu_r \\ &= \frac{|\alpha| - L - (|s| + 3 - L)^+ - (|r| - L)^+}{2} \\ &\geq \frac{|\alpha| - |s| - |r| - 3 + L}{2} \\ &= \frac{L - 3}{2} \geq 0 \end{aligned}$$

as long as $L \geq 3$. By assumption, $L \geq 2$. If $L = 2$, then $|\alpha| = 3$. In that case, we cannot have $|s| \leq |\alpha| - 4$. Therefore, we conclude that $M \geq 0$. Further,

$$||t^{\nu_s}\partial^s u_x||_{L^\infty} \leq C\left(\int t^{2\nu_s}[(\partial^s u_x)^2 + (\partial^s u_{xxx})^2 + (\partial^s u_{xy})^2]\right)^{1/2}.$$

Each of those terms is of order at most $|\alpha| - 1$, and therefore, bounded by terms of the form (5.2). Further, $2L - |\alpha| - \alpha_2 \leq 2L - |r| - r_2$. Therefore, for $|r| \leq |\alpha| - 1$, the last term above is bounded by terms of the form (5.2). In the case that $|r| = |\alpha|$, we have $|s| = 0$, and therefore

$$\begin{aligned}\left|\int_0^T \int f_\alpha(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\right| &\leq C||u_x||_{L^\infty} \int_0^T \int f_\alpha(\partial^\alpha u)^2 \\ &\leq C||u||_{X^0} \int_0^T \int f_\alpha(\partial^\alpha u)^2.\end{aligned}$$

Combining our estimates above on

$$\left|\int_0^T \int f_\alpha(\partial^\alpha u)(\partial^r u)(\partial^s u_x)\right|$$

with (5.7), we see that

$$\int f_\alpha(\partial^\alpha u)^2 + 3\int_0^T \int (f_\alpha)_x(\partial^\alpha u_x)^2 \leq C + C\sum_{\substack{|\gamma|=|\alpha| \\ 2L-|\gamma|-\gamma_2\geq 1}} \int_0^T \int f_\gamma(\partial^\gamma u)^2 \tag{5.9}$$

where the constant C depends only on terms of the form (5.2) and (5.3).

Using the above estimate for all derivatives γ of order $|\alpha|$ such that $2L - |\gamma| - \gamma_2 \geq 1$, we see that

$$\begin{aligned}\sum_{\substack{|\gamma|=|\alpha| \\ 2L-|\gamma|-\gamma_2\geq 1}} &\left[\int f_\gamma(\partial^\alpha u)^2 + 3\int_0^T \int (f_\gamma)_x(\partial^\alpha u_x)^2\right] \\ &\leq C + \sum_{\substack{|\gamma|=|\alpha| \\ 2L-|\gamma|-\gamma_2\geq 1}} \int_0^T \int f_\gamma(\partial^\gamma u)^2.\end{aligned} \tag{5.10}$$

where C depends only on terms of the form (5.2) and (5.3). Applying Gronwall's inequality, we get the desired estimate. □

6 Persistence Theorem

In section four we proved the existence of a solution u to (2.1) in $L^\infty([0,T]; X^N(\mathbb{R}^2))$ for given initial data $\phi \in X^N(\mathbb{R}^2)$. In this section, we prove that if, in addition, our initial data

ϕ lies in the weighted space $\widetilde{H}_x^K(W_{0\,K\,0})$ for some $K \geq 0$, then the solution u also lies in $L^\infty([0,T]; \widetilde{H}_x^K(W_{0\,K\,0}))$. This property is known as a "persistence" property of the initial data. This property provides a basis for starting the induction in our Gain of Regularity theorem in Section 7.

Theorem 6.1 *Suppose $u \in L^\infty([0,T] : X^1(\mathbb{R}^2))$ with initial data $\phi(x,y) \in X^1(\mathbb{R}^2)$ such that ϕ also lies in $\widetilde{H}_x^K(W_{0\,K\,0})$ for some integer $K \geq 0$. Then*

$$u \in L^\infty([0,T] : X^1(\mathbb{R}^2) \cap \widetilde{H}_x^K(W_{0\,K\,0}))$$

and

$$\sup_{0 \leq t \leq T} \int f_{\alpha_1}(\partial^\alpha u)^2 + \int_0^T \int g_{\alpha_1}(\partial^\alpha u_x)^2 \, dt \leq C$$

for $|\alpha| \leq K$, $\alpha_1 \neq 0$, where $f_{\alpha_1} \in W_{0\,\alpha_1\,0}$ and $g_{\alpha_1} \in W_{\sigma\,\alpha_1 - 1\,0}$ for $\sigma > 0$ arbitrary and C depends only on T and the norm of $\phi \in X^1(\mathbb{R}^2) \cap \widetilde{H}_x^K(W_{0\,K\,0})$.

Proof. We use induction on $j = |\alpha|$ for $1 \leq j \leq K$. The case that $j = 0$ follows from conservation of L^2 norm. We derive formally some a priori estimates for the solution where the bound involves only the norms of $u \in L^\infty([0,T] : X^1(\mathbb{R}^2))$ and the norms of $\phi \in \widetilde{H}_x^K(W_{0\,K\,0})$. Then, we can apply convergence arguments to show that the result holds true for general solutions. In order to do so, we need to approximate general solutions $u \in X^1(\mathbb{R}^2)$ by smooth solutions and approximate general weight functions $f \in W_{0\,j\,0}$ by smooth, bounded weight functions. The first of these procedures has already been discussed, so we will concentrate on the second.

For a fixed i, we begin by taking a sequence of bounded weight functions $g_{i,\delta}$ which decay as $|x| \to \infty$ and which approximate $g_i \in W_{\sigma\,i-1\,0}$ with $\sigma > 0$ from below, uniformly on any half-line $(-\infty, c)$. Define the weight functions

$$f_{i,\delta}(x,t) = 1 + \int_{-\infty}^x g_{i,\delta}(z,t)\, dz.$$

Therefore, the functions $f_{i,\delta}$ are bounded weight functions approximating $f_i \in W_{0,i,0}$ from below, uniformly on compact sets.

¿From (5.3) and using the fact that $\partial_t(f_{i,\delta}) \leq c f_{i,\delta}$ and $\partial_x(f_{i,\delta}) \leq c f_{i,\delta}$, we have

$$\int f_{i,\delta}(\partial^\alpha u)^2 + 3\int_0^T \int (f_{i,\delta})_x (\partial^\alpha u_x)^2 \, dt \leq \int_0^T (f_{i,\delta})_x (\partial^\alpha \partial_x^{-1} u_y)^2 \, dt$$
$$+ C \int_0^T \int f_{i,\delta}(\partial^\alpha u)^2 \, dt + 2\left| \int_0^T \int f_{i,\delta}(\partial^\alpha u)\partial^\alpha(u u_x)\, dt \right|.$$

The case $j = 1$.

(a) **The subcase $\alpha = (1,0)$.** Defining $g_{1,\delta}$ and $f_{1,\delta}$ as above, we see that $f_{1,\delta}$ will approximate $f_1 \in W_{0\,1\,0}$ from below. Differentiating (2.1) in the $x-$variable, multiplying by

$2f_{1,\delta}$ and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}
&\partial_t \int f_{1,\delta} u_x^2 + 3 \int (f_{1,\delta})_x u_{xx}^2 \\
&\quad = \int (f_{1,\delta})_x u_y^2 + \int [\partial_t f_{1,\delta} + \partial_x^3 f_{1,\delta} + \partial_x f_{1,\delta}] u_x^2 - 2 \int f_{1,\delta} u_x (u u_x)_x \qquad (6.1)\\
&\le C \int u_y^2 + C \int f_{1,\delta} u_x^2 + 2 \left| \int f_{1,\delta} u_x (u u_x)_x \right|
\end{aligned}$$

Moreover

$$\begin{aligned}
\left| \int f_{1,\delta} u_x (u u_x)_x \right| &= \left| \int f_{1,\delta} u_x [u_x^2 + u u_{xx}] \right| \\
&= \left| \int f_{1,\delta} [u_x^3 + u u_x u_{xx}] \right| \\
&\le c ||u_x||_{L^\infty} \int f_{1,\delta} u_x^2 + \frac{1}{2} \left| \int (f_{1,\delta} u)_x u_x^2 \right| \\
&\le c(||u||_{L^\infty} + ||u_x||_{L^\infty}) \int f_{1,\delta} u_x^2 \\
&\le C ||u||_{X^0} \int f_{1,\delta} u_x^2.
\end{aligned}$$

Combining this estimate with (6.1), we conclude that for $0 \le t \le T$,

$$\int f_{1,\delta}(\cdot, t) u_x^2 + 3 \int_0^T \int (f_{1,\delta})_x u_{xx}^2 \le C \int f_{1,\delta}(\cdot, 0) \phi_x^2 + C \int_0^T \int u_y^2 + C \int_0^T \int f_{1,\delta} u_x^2.$$

Applying Gronwall's inequality, we conclude that

$$\sup_{0 \le t \le T} \int f_{1,\delta} u_x^2 + 3 \int_0^T \int (f_{1,\delta})_x u_{xx}^2 \le C$$

where C does not depend on δ but only on T and the norm of $\phi \in X^1(\mathbb{R}^2) \cap \widetilde{H}_x^1(W_{0\,1\,0})$. Taking the limit as $\delta \to \infty$, we conclude that

$$\sup_{0 \le t \le T} \int f_1 u_x^2 + 3 \int_0^T \int g_1 u_{xx}^2 \le C, \qquad (6.2)$$

as claimed.

(b) **The subcase** $\alpha = (0, 1)$**.** Here, our weight function $f_0 \in W_{0\,0\,0}$. Differentiating (2.1) in the y-variable, multiplying by $2 f_{0,\delta} u_y$ and integrating over $\mathbb{R}^2$, we have

$$2 \int f_{0,\delta} u_y u_{yt} + 2 \int f_{0,\delta} u_y u_{xxxy} - 2 \int f_{0,\delta} u_y \partial_x^{-1} u_{yyy} + 2 \int f_{0,\delta} u_y u_{xy} + 2 \int f_{0,\delta} u_y (u u_x)_y = 0.$$

Integrating each term by parts gets

$$
\begin{aligned}
&\partial_t \int f_{0,\delta} u_y^2 + 3\int (f_{0,\delta})_x u_{xy}^2 \\
&\qquad \le \int (f_{0,\delta})_x (\partial_x^{-1} u_{yy})^2 + C\int f_{0,\delta} u_y^2 + 2\left|\int f_{0,\delta} u_y (uu_x)_y\right| \\
&\le C\int (\partial_x^{-1} u_{yy})^2 + C\int f_{0,\delta} u_y^2 + 2\left|\int f_{0,\delta} u_y (uu_x)_y\right|.
\end{aligned} \tag{6.3}
$$

Moreover,

$$
\begin{aligned}
\left|\int f_{0,\delta} u_y (uu_x)_y\right| &= \left|\int f_{0,\delta} u_y (u_y u_x + uu_{xy})\right| \\
&\le C||u_x||_{L^\infty} \int f_{0,\delta} u_y^2 + C||u||_{L^\infty} \int (f_{0,\delta})_x u_y^2 \\
&\le C||u||_{X^0} \int f_{0,\delta} u_y^2.
\end{aligned}
$$

Combining this estimate with (6.3), we conclude that for $0 \le t \le T$,

$$
\int f_{0,\delta}(\cdot,t) u_y^2 + 3\int_0^T \int (f_{0,\delta})_x u_{xy}^2 \le \int f_{0,\delta}(\cdot,0)\phi_y^2 + C\int_0^T \int (\partial_x^{-1} u_{yy})^2 + C\int_0^T \int f_{0,\delta} u_y^2. \tag{6.4}
$$

Applying Gronwall's inequality, we conclude that

$$
\sup_{0\le t\le T} \int f_{0,\delta} u_y^2 + 3\int_0^T \int (f_{0,\delta})_x u_{xy}^2 \le C
$$

where C does not depend on δ, but only on T and the norm of $\phi \in X^0(\mathbb{R}^2) \cap \widetilde{H}_x^1(W_{0\,1\,0})$. Passing to the limit, we conclude that

$$
\sup_{0\le t\le T} \int f_0 u_y^2 + 3\int_0^T \int g_0 u_{xy}^2 \le C. \tag{6.5}
$$

The case $j = 2$.

(a) **The subcase $\alpha = (2, 0)$.** In this case, $f_{2,\delta}$ will approximate $f_2 \in W_{0\,2\,0}$. In a similar way as above, we have

$$
\begin{aligned}
&2\int f_{2,\delta} u_{xx} u_{xxt} + 2\int f_{2,\delta} u_{xx} u_{xxxxx} - 2\int f_{2,\delta} u_{xx} u_{xyy} \\
&\qquad + 2\int f_{2,\delta} u_{xx} u_{xxx} + 2\int f_{2,\delta} u_{xx} (uu_x)_{xx} = 0.
\end{aligned}
$$

Integrating each term by parts gets

$$
\begin{aligned}
&\partial_t \int f_{2,\delta} u_{xx}^2 + 3\int (f_{2,\delta})_x u_{xxx}^2 \\
&\qquad = c\int (f_{2,\delta})_x u_{xy}^2 + \int [\partial_t f_{2,\delta} + \partial_x^3 f_{2,\delta} + \partial_x f_{2,\delta}] u_{xx}^2 - 2\int f_{2,\delta} u_{xx} (uu_x)_{xx} \\
&\le \int f_{2,\delta} u_{xy}^2 + c\int f_{2,\delta} u_{xx}^2 + 2\left|\int f_{2,\delta} u_{xx} (uu_x)_{xx}\right|.
\end{aligned}
$$

Moreover

$$\begin{aligned}\left|\int f_{2,\delta}u_{xx}(uu_x)_{xx}\right| &= \left|\int f_{2,\delta}u_{xx}[3u_xu_{xx}+uu_{xxx}]\right| \\ &= \left|\int f_{2,\delta}[3u_xu_{xx}^2+uu_{xx}u_{xxx}]\right| \\ &\le c||u_x||_{L^\infty}\int f_{2,\delta}u_{xx}^2+\frac{1}{2}\left|(f_{2,\delta}u)_xu_{xx}^2\right| \\ &\le c(||u||_{L^\infty}+||u_x||_{L^\infty})\int f_{2,\delta}u_{xx}^2 \\ &\le C||u||_{X^0}\int f_{2,\delta}u_{xx}^2.\end{aligned}$$

Therefore,

$$\partial_t\int f_{2,\delta}u_{xx}^2+3\int(f_{2,\delta})_xu_{xxx}^2\le\int(f_{2,\delta})_xu_{xy}^2+C\int f_{2,\delta}u_{xx}^2,$$

where C depends only on the norm of $\phi\in X^0(\mathbb{R}^2)$. We will combine this estimate with the estimate below.

(b) **The subcase** $\alpha=(1,1)$ Applying $\partial_x\partial_y$ to (2.1), multiplying by $2f_{1,\delta}u_{xy}$ where $f_{1,\delta}$ approximates $f_1\in W_{0\,1\,0}$, and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}&2\int f_{1,\delta}u_{xy}u_{xyt}+2\int f_{1,\delta}u_{xy}u_{xxxxy}-2\int f_{1,\delta}u_{xy}u_{yyy} \\ &\quad+2\int f_{1,\delta}u_{xy}u_{xxy}+2\int f_{1,\delta}u_{xy}(uu_x)_{xy}=0.\end{aligned}$$

Integrating each term by parts gets

$$\begin{aligned}&\partial_t\int f_{1,\delta}u_{xy}^2+3\int(f_{1,\delta})_xu_{xxy}^2 \\ &\qquad=\int(f_{1,\delta})_xu_{yy}^2+\int[\partial_tf_{1,\delta}+\partial_x^3f_{1,\delta}+\partial_xf_{1,\delta}]u_{xy}^2-2\int f_{1,\delta}u_{xy}(uu_x)_{xy} \\ &\le\int(f_{1,\delta})_xu_{yy}^2+c\int f_{1,\delta}u_{xy}^2+2\left|\int f_{1,\delta}u_{xy}(uu_x)_{xy}\right|.\end{aligned}$$

Moreover,

$$\begin{aligned}\left|\int f_{1,\delta}u_{xy}(uu_x)_{xy}\right| &= \left|\int f_{1,\delta}u_{xy}(2u_xu_{xy}+u_{xx}u_y+uu_{xxy})\right| \\ &\le C(||u_x||_{L^\infty}+||u||_{L^\infty})\int f_{1,\delta}u_{xy}^2+\left|\int f_{1,\delta}u_{xx}u_{xy}u_y\right| \\ &\le C||u||_{X^0}\int f_{1,\delta}u_{xy}^2+C||u_y||_{L^\infty}(\int f_{1,\delta}u_{xx}^2+\int f_{1,\delta}u_{xy}^2) \\ &\le C||u||_{X^0}\int f_{1,\delta}u_{xy}^2+C||u||_{X^1}\int f_{1,\delta}(u_{xx}^2+u_{xy}^2)\end{aligned}$$

since

$$||u_y||_{L^\infty} \leq \left(\int u_y^2 + u_{xxy}^2 + u_{yy}^2\right)^{1/2} \leq ||u||_{X^1}.$$

Therefore,

$$\partial_t \int f_{1,\delta} u_{xy}^2 + 3 \int (f_{1,\delta})_x u_{xxy}^2 \leq \int (f_{1,\delta})_x u_{yy}^2 + C \int f_{1,\delta} u_{xx}^2 + C \int f_{1,\delta} u_{xy}^2.$$

(c) **The subcase** $\alpha = (0, 2)$**.** Applying ∂_y^2 to (2.1), multiplying by u_{yy} and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned} 2\int u_{yy} u_{yyt} + 2\int u_{yy} u_{xxxyy} - 2\int u_{yy} \partial_x^{-1} u_{yyyy} \\ + 2\int u_{yy} u_{xyy} + 2\int u_{yy} (u u_x)_{yy} = 0. \end{aligned}$$

Integrating by parts gets

$$\partial_t \int u_{yy}^2 \leq 2\left|\int u_{yy}(uu_x)_{yy}\right|.$$

Now

$$\begin{aligned} \left|\int u_{yy}(uu_x)_{yy}\right| &= \left|\int u_{yy}(u_{yy}u_x + 2u_y u_{xy} + u u_{xyy})\right| \\ &\leq ||u_x||_{L^\infty} \int u_{yy}^2 + ||u_y||_{L^\infty} \left(\int u_{yy}^2 + u_{xy}^2\right) \\ &\leq ||u||_{X^0} \int u_{yy}^2 + C||u||_{X^1} \int (u_{yy}^2 + u_{xy}^2). \end{aligned}$$

Now combining these estimates from (a), (b) and (c) above, we have

$$\begin{aligned} &\partial_t \int (f_{2,\delta} u_{xx}^2 + f_{1,\delta} u_{xy}^2 + u_{yy}^2) + 3\int \left[(f_{2,\delta})_x u_{xxx}^2 + (f_{1,\delta})_x u_{xxy}^2\right] \\ &\quad \leq C\int (f_{2,\delta} + f_{1,\delta}) u_{xx}^2 + C\int ((f_{2,\delta})_x + f_{1,\delta} + 1) u_{xy}^2 + \int ((f_{1,\delta})_x + 1) u_{yy}^2, \end{aligned}$$

where C depends only on the norm of $\phi \in X^1(\mathbb{R}^2)$. Since $f_{2,\delta}$ approximates $f_2 \in W_{0\,2\,0}$ and $f_{1,\delta}$ approximates $f_1 \in W_{0\,1\,0}$, we can choose $f_{2,\delta}, f_{1,\delta}$ such that $(f_{2,\delta})_x \leq C f_{1,\delta}$, etc. Therefore,

$$\begin{aligned} &\partial_t \int (f_{2,\delta} u_{xx}^2 + f_{1,\delta} u_{xy}^2 + u_{yy}^2) + 3\int \left[(f_{2,\delta})_x u_{xxx}^2 + (f_{1,\delta})_x u_{xxy}^2\right] \\ &\quad \leq C\int (f_{2,\delta} u_{xx}^2 + f_{1,\delta} u_{xy}^2 + u_{yy}^2). \end{aligned}$$

Integrating with respect to t, we have

$$\int (f_{2,\delta}(\cdot,t)u_{xx}^2 + f_{1,\delta}(\cdot,t)u_{xy}^2 + u_{yy}^2) + 3\int_0^t \int \left[(f_{2,\delta})_x u_{xxx}^2 + (f_{1,\delta})_x u_{xxy}^2\right]$$
$$\leq \int (f_{2,\delta}(\cdot,0)\phi_{xx}^2 + f_{1,\delta}(\cdot,0)\phi_{xy}^2 + \phi_{yy}^2) + C\int_0^t \int (f_{2,\delta}u_{xx}^2 + f_{1,\delta}u_{xy}^2 + u_{yy}^2).$$

Further, integrating by parts and using the fact that f_1 approximates $f_{1,\delta} \approx x$ for $x > 1$, we note that

$$\int f_{1,\delta}\phi_{xy}^2 \leq C\int f_{2,\delta}\phi_{xx}^2 + C\int \phi_{yy}^2.$$

Therefore, by Gronwall's inequality

$$\sup_{0\leq t\leq T} \int (f_{2,\delta}(\cdot,t)u_{xx}^2 + f_{1,\delta}(\cdot,t)u_{xy}^2 + u_{yy}^2) + 3\int_0^T \int \left[(f_{2,\delta})_x u_{xxx}^2 + (f_{1,\delta})_x u_{xxy}^2\right] \leq C$$

where C does not depend on δ but only on T and the norm of $\phi \in X^1(\mathbb{R}^2) \cap \widetilde{H}_x^2(W_{0\,2\,0})$. Consequently, we can pass to the limit and conclude that

$$\sup_{0\leq t\leq T} \int (f_2(\cdot,t)u_{xx}^2 + f_1(\cdot,t)u_{xy}^2 + u_{yy}^2) + 3\int_0^T \int \left[g_2 u_{xxx}^2 + g_1 u_{xxy}^2\right] \leq C.$$

The case $j = 3$.

(a) **The subcase $\alpha = (3, 0)$.** We choose our weight functions such that $f_{3,\delta}$ approximates $f_3 \in W_{0\,3\,0}$. Applying ∂_x^3 to (2.1), multiplying by $f_{3,\delta}u_{xxx}$ and integrating over $\mathbb{R}^2$, we have

$$2\int f_{3,\delta}u_{xxx}u_{xxxt} + 2\int f_{3,\delta}u_{xxx}u_{xxxxxx} - 2\int f_{3,\delta}u_{xxx}u_{xxyy}$$
$$+ 2\int f_{3,\delta}u_{xxx}u_{xxxx} + 2\int f_{3,\delta}u_{xxx}(uu_x)_{xxx} = 0.$$

Integrating by parts gets

$$\partial_t \int f_{3,\delta}u_{xxx}^2 + 3\int (f_{3,\delta})_x u_{xxxx}^2$$
$$= \int (f_{3,\delta})_x u_{xxy}^2 + \int [\partial_t f_{3,\delta} + \partial_x^3 f_{3,\delta} + \partial_x f_{3,\delta}]u_{xxx}^2 - 2\int f_{3,\delta}u_{xxx}(uu_x)_{xxx}$$
$$\leq \int (f_{3,\delta})_x u_{xxy}^2 + c\int f_{3,\delta}u_{xxx}^2 + 2\left|\int f_{3,\delta}u_{xxx}(uu_x)_{xxx}\right|.$$

Moreover

$$\begin{aligned}
\left|\int f_{3,\delta}u_{xxx}(uu_x)_{xxx}\right| &= \left|\int f_{3,\delta}[3u_{xx}^2 + 4u_xu_{xxx} + uu_{xxxx}]u_{xxx}\right| \\
&\leq 3\left|\int f_{3,\delta}u_{xx}^2u_{xxx}\right| + 4\left|\int f_{3,\delta}u_xu_{xxx}^2\right| + \left|\int f_{3,\delta}uu_{xxx}u_{xxxx}\right| \\
&\leq 3\left|\int f_{3,\delta}u_{xx}^2u_{xxx}\right| + c||u_x||_{L^\infty}\int f_{3,\delta}u_{xxx}^2 + c\left|\int (f_{3,\delta}u)_xu_{xxx}^2\right| \\
&\leq 3\left|\int f_{3,\delta}u_{xx}^2u_{xxx}\right| + c||u||_{X^0}\int f_{3,\delta}u_{xxx}^2 \\
&\quad + c(||u||_{L^\infty(\mathbb{R}^2)} + ||u_x||_{L^\infty(\mathbb{R}^2)})\int f_{3,\delta}u_{xxx}^2 \\
&\leq C\left|\int f_{3,\delta}u_{xx}^2u_{xxx}\right| + c||u||_{X^0(\mathbb{R}^2)}\int f_{3,\delta}u_{xxx}^2.
\end{aligned}$$

Now we estimate the first term on the right-hand side.

Case: $x > 1$. Let $A_1 = \{x \in \mathbb{R} : x > 1\} \times \mathbb{R} \subseteq \mathbb{R}^2$.

$$\begin{aligned}
\left|\int_{A_1} f_{3,\delta}u_{xx}^2u_{xxx}\right| &= C\left|\int_{A_1}(f_{3,\delta})_xu_{xx}^3\right| \\
&\leq C||u_{xx}||_{L^\infty(A_1)}\int_{A_1}(f_{3,\delta})_xu_{xx}^2 \\
&\leq C\left(\int_{A_1}u_{xx}^2 + u_{xxxx}^2 + u_{xxy}^2\right)^{1/2}\int_{A_1}f_{2,\delta}u_{xx}^2 \\
&\leq \epsilon\int_{A_1}(u_{xx}^2 + u_{xxxx}^2 + u_{xxy}^2) + C\left(\int_{A_1}f_{2,\delta}u_{xx}^2\right)^2.
\end{aligned}$$

Now the terms involving u_{xx} have been bounded by the previous step in the induction. Therefore, we conclude that

$$\left|\int_{A_1} f_{3,\delta}u_{xx}^2u_{xxx}\right| \leq C + \epsilon\int_{A_1}(u_{xxxx}^2 + u_{xxy}^2).$$

Case: $x < -1$. Let $A_{-1} = \{x \in \mathbb{R} : x < -1\} \times \mathbb{R} \subseteq \mathbb{R}^2$. We use the fact that $f_{3,\delta} \approx c$ to show

$$\begin{aligned}
\left|\int_{A_{-1}} f_{3,\delta}u_{xx}^2u_{xxx}\right| &\leq c\left|\int_{A_{-1}}u_{xx}^2u_{xxx}\right| \\
&\leq c\left(\int_{A_{-1}}u_{xx}^4\right)^{1/2}\left(\int_{A_{-1}}u_{xxx}^2\right)^{1/2} \\
&\leq C\left(\int_{A_{-1}}[u_{xx}^2 + u_{xxx}^2 + u_{xy}^2]\right)\left(\int_{A_{-1}}u_{xxx}^2\right)^{1/2} \\
&\leq c||u||_{X^0}^3.
\end{aligned}$$

Combining these estimates for the subcase $\alpha = (3,0)$, yields

$$\begin{aligned}
&\int f_{3,\delta}(\cdot,t)u_{xxx}^2 + 3\int_0^t\int (f_{3,\delta})_x u_{xxxx}^2 \\
&\le \int f_{3,\delta}(\cdot,0)\phi_{xxx}^2 + C + \int_0^t\int (f_{3,\delta})_x u_{xxy}^2 + c\int_0^t\int f_{3,\delta}u_{xxx}^2 + \epsilon\int_0^t\int_{A_1} u_{xxxx}^2 \\
&\le \int f_{3,\delta}(\cdot,0)\phi_{xxx}^2 + C + \int_0^t\int (f_{3,\delta})_x u_{xxy}^2 + c\int_0^t\int f_{3,\delta}u_{xxx}^2 + \epsilon\int_0^t\int (f_{3,\delta})_x u_{xxxx}^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int f_{3,\delta}(\cdot,t)u_{xxx}^2 + 3\int_0^t\int (f_{3,\delta})_x u_{xxxx}^2 \\
&\le \int f_{3,\delta}(\cdot,0)\phi_{xxx}^2 + C + \int_0^t\int (f_{3,\delta})_x u_{xxy}^2 + c\int_0^t\int f_{3,\delta}u_{xxx}^2 \\
&\le \int f_{3,\delta}(\cdot,0)\phi_{xxx}^2 + C + C\int_0^t\int f_{2,\delta}u_{xxy}^2 + C\int_0^t\int f_{3,\delta}u_{xxx}^2.
\end{aligned}$$

(b) **The subcase $\alpha = (2,1)$.** In this case, we take $f_{2,\delta}$ approximating $f_2 \in W_{0\,2\,0}$. Apply $\partial_x^2\partial_y$ to (2.1), multiply by $f_{2,\delta}u_{xxy}$ and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}
&\partial_t \int f_{2,\delta}u_{xxy}^2 + 3\int (f_{2,\delta})_x u_{xxxy}^2 \\
&\le \int (f_{2,\delta})_x u_{xyy}^2 + \int f_{2,\delta}u_{xxy}^2 + 2\left|\int f_{2,\delta}u_{xxy}(uu_x)_{xyy}\right|.
\end{aligned}$$

Now

$$\begin{aligned}
\left|\int f_{2,\delta}u_{xxy}(uu_x)_{xxy}\right| &= \left|\int f_{2,\delta}u_{xxy}[3u_{xy}u_{xx} + 3u_xu_{xxy} + u_yu_{xxx} + uu_{xxxy}]\right| \\
&\le C\int f_{2,\delta}u_{xxy}u_{xy}u_{xx} + \int f_{2,\delta}u_{xxy}u_yu_{xxx} + C||u||_{X^0}\int f_{2,\delta}u_{xxy}^2.
\end{aligned}$$

The second term on the right-hand side satisfies

$$\begin{aligned}
\int f_{2,\delta}u_{xxy}u_yu_{xxx} &\le C||u_y||_{L^\infty}\int f_{2,\delta}(u_{xxx}^2 + u_{xxy}^2) \\
&\le C||u||_{X^1}\int (f_{3,\delta}u_{xxx}^2 + f_{2,\delta}u_{xxy}^2).
\end{aligned}$$

For the first term on the right-hand side, we consider two cases. First, for A_1,

$$\begin{aligned}
\int_{A_1} f_{2,\delta}u_{xxy}u_{xy}u_{xx} &\le ||u_{xy}||_{L^\infty(A_1)}\left(\int_{A_1} f_{2,\delta}u_{xxy}^2\right)^{1/2}\left(\int_{A_1} f_{2,\delta}u_{xx}^2\right)^{1/2} \\
&\le \epsilon\left(\int_{A_1} u_{xy}^2 + u_{xxxy}^2 + u_{xyy}^2\right) + C\int_{A_1} f_{2,\delta}u_{xxy}^2 \\
&\le C + \epsilon\int (f_{2,\delta})_x u_{xxxy}^2 + C\int f_{1,\delta}u_{xyy}^2 + C\int f_{2,\delta}u_{xxy}^2.
\end{aligned}$$

Then for A_{-1},

$$\begin{aligned}\int_{A_{-1}} f_{2,\delta} u_{xxy} u_{xy} u_{xx} &= \int_{A_{-1}} u_{xxy} u_{xy} u_{xx} \\ &= C \int_{A_{-1}} u_{xy}^2 u_{xxx} \\ &\leq C \left(\int_{A_{-1}} u_{xy}^4 \right)^{1/2} \left(\int_{A_{-1}} u_{xxx}^2 \right)^{1/2} \\ &\leq C \left(\int_{A_{-1}} u_{xy}^2 + u_{xxy}^2 + u_{yy}^2 \right) \left(\int_{A_{-1}} u_{xxx}^2 \right)^{1/2} \\ &\leq C(||u||_{X^0}) \left(1 + \int f_{2,\delta} u_{xxy}^2 \right)\end{aligned}$$

Combining these estimates and integrating with respect to t, we have

$$\begin{aligned}\int f_{2,\delta}(\cdot,t) u_{xxy}^2 + 3 \int_0^t \int (f_{2,\delta})_x u_{xxxy}^2 \leq C + \int f_{2,\delta}(\cdot,0) \phi_{xxy}^2 + \epsilon \int_0^t \int (f_{2,\delta})_x u_{xxxy}^2 \\ + \int_0^t \int f_{2,\delta} u_{xxy}^2 + C \int_0^t \int f_{1,\delta} u_{xyy}^2.\end{aligned}$$

Therefore,

$$\int f_{2,\delta}(\cdot,t) u_{xxy}^2 + 3 \int_0^t \int (f_{2,\delta})_x u_{xxxy}^2 \leq C + \int f_{2,\delta}(\cdot,0) \phi_{xxy}^2 + \int_0^t \int f_{2,\delta} u_{xxy}^2 + C \int_0^t \int f_{1,\delta} u_{xyy}^2.$$

(c) **The subcase** $\alpha = (1,2)$**.** We take our weight function $f_{1,\delta}$ approximating $f_1 \in W_{0\,1\,0}$. Applying $\partial_x \partial_y^2$ to (2.1), multiplying by $f_{1,\delta} u_{xyy}$ and integrating over $\mathbb{R}^2$, we get

$$\begin{aligned}&\partial_t \int f_{1,\delta} u_{xyy}^2 + 3 \int (f_{1,\delta})_x u_{xxyy}^2 \\ &\leq c \int (f_{1,\delta})_x u_{yyy}^2 + \int f_{1,\delta} u_{xyy}^2 + 2 \left| \int f_{1,\delta} u_{xyy} (u u_x)_{xyy} \right|.\end{aligned}$$

Now

$$2 \left| \int f_{1,\delta} u_{xyy} (u u_x)_{xyy} \right| = 2 \left| \int f_{1,\delta} u_{xyy} (2u_{xy}^2 + 2u_x u_{xyy} + u_{yy} u_{xx} + 2u_y u_{xxy} + u u_{xxyy}) \right|.$$

The first term on the right-hand side satisfies

$$\int f_{1,\delta} u_{xyy} u_{xy}^2 = C \int f_{1,\delta} (u_{xy}^3)_y = 0,$$

since $(f_{1,\delta})_y = 0$. Integrating by parts, it is clear that the second and fifth terms on the right-hand side are bounded by

$$C||u||_{X^0} \int f_{1,\delta} u_{xyy}^2.$$

The fourth term on the right-hand side is bounded by

$$C||u_y||_{L^\infty}\left(\int f_{1,\delta}u_{xyy}^2+\int f_{1,\delta}u_{xxy}^2\right).$$

For the third-term on the right-hand side, we consider the cases when $x > 1$ and $x < -1$ separately. First, for $x > 1$, we have

$$\begin{aligned}\int_{A_1} f_{1,\delta}u_{xyy}u_{yy}u_{xx} &\le ||u_{yy}||_{L^\infty(A_1)}\left(\int_{A_1} f_{1,\delta}u_{xyy}^2\right)^{1/2}\left(\int_{A_1} f_{1,\delta}u_{xx}^2\right)^{1/2}\\ &\le C\left(\int_{A_1} u_{yy}^2+u_{xxyy}^2+u_{yyy}^2\right)^{1/2}\left(\int_{A_1} f_{1,\delta}u_{xyy}^2\right)^{1/2}\\ &\le C+\epsilon\int_{A_1} u_{xxyy}^2+C\int_{A_1} f_{1,\delta}u_{xyy}^2+C\int_{A_1} u_{yyy}^2\end{aligned}$$

where we have used the fact that $\int f_{1,\delta}u_{xx}^2$ was bounded on the previous step of the induction. We will bound the ϵ term back on the left-hand side. For $x < -1$, we have

$$\begin{aligned}\int_{A_{-1}} f_{1,\delta}u_{xyy}u_{yy}u_{xx} &\approx \int \chi_{[x<-1]}u_{yy}^2u_{xxx}\\ &\le \left(\int_{A_{-1}} u_{yy}^4\right)^{1/2}\left(\int_{A_{-1}} u_{xxx}^2\right)^{1/2}\\ &\le C\int_{A_{-1}} u_{yy}^2+u_{xyy}^2+(\partial_x^{-1}u_{yy})^2\\ &\le C\end{aligned}$$

where C depends only on the norm of $u \in X^1(\mathbb{R}^2)$. Combining these estimates and integrating with respect to t, we have

$$\int f_{1,\delta}(\cdot,t)u_{xyy}^2+3\int_0^t\int (f_{1,\delta})_x u_{xxyy}^2 \le \int f_{1,\delta}(\cdot,0)\phi_{xyy}^2+C\int_0^t\int f_{1,\delta}u_{xyy}^2+C\int_0^t\int u_{yyy}^2.$$

(d) **The subcase $\alpha = (0,3)$.** In this case we apply ∂_y^3 to (2.1), multiply by u_{yyy} and integrate over $\mathbb{R}^2$. We have

$$\partial_t\int u_{yyy}^2 \le \int u_{yyy}^2+2\left|\int u_{yyy}(uu_x)_{yyy}\right|.$$

Now

$$\left|\int u_{yyy}(uu_x)_{yyy}\right| = \left|\int u_{yyy}(u_xu_{yyy}+3u_{yy}u_{xy}+3u_yu_{xyy}+uu_{xyyy})\right|$$

Integrating by parts as necessary, we see that the first and fourth terms on the right-hand side are bounded by

$$||u||_{X^0}\int u_{yyy}^2.$$

The second term on the right-hand side is bounded by

$$
\begin{aligned}
&\left(\int u_{yy}^4\right)^{1/4}\left(\int u_{xy}^4\right)^{1/4}\left(\int u_{yyy}^2\right)^{1/2}\\
&\le \left(\int u_{yy}^2+u_{xyy}^2+(\partial_x^{-1}u_{yyy})^2\right)^{1/2}\left(\int u_{xy}^2+u_{xxy}^2+u_{yy}^2\right)^{1/2}\left(\int u_{yyy}^2\right)^{1/2}\\
&\le C||u||_{X^1(\mathbb{R}^2)}\left(\int u_{yyy}^2\right)^{1/2}\\
&\le C+C\int u_{yyy}^2.
\end{aligned}
$$

The third term on the right-hand side is bounded by

$$
\begin{aligned}
&C||u_y||_{L^\infty}\left(\int u_{xyy}^2\right)^{1/2}\left(\int u_{yyy}^2\right)^{1/2}\\
&\le C\left(\int u_y^2+u_{xxy}^2+u_{yy}^2\right)^{1/2}\left(\int u_{xyy}^2\right)^{1/2}\left(\int u_{yyy}^2\right)^{1/2}\\
&\le C||u||_{X^1}^2\left(\int u_{yyy}^2\right)^{1/2}\\
&\le C+C\int u_{yyy}^2.
\end{aligned}
$$

Combining the estimates above and integrating with respect to t, we have

$$
\int u_{yyy}^2 \le \int \phi_{yyy}^2 + C + C\int_0^t\int u_{yyy}^2
$$

where C depends only on the norm of ϕ in $X^1(\mathbb{R}^2)$.

Combining the estimates above for (a), (b), (c) and (d) and applying Gronwall's inequality, we conclude that

$$
\begin{aligned}
&\sup_{0\le t\le T}\int (f_{3,\delta}u_{xxx}^2+f_{2,\delta}u_{xxy}^2+f_{1,\delta}u_{xyy}^2+u_{yyy}^2)\\
&+3\int_0^T\int ((f_{3,\delta})_x u_{xxxx}^2+(f_{2,\delta})_x u_{xxxy}^2+(f_{1,\delta})_x u_{xxyy}^2)\le C
\end{aligned}
$$

where C does not depend on δ, but only on T and the norm of $\phi\in X^1(\mathbb{R}^2)\cap \widetilde{H}_x^3(W_{0\,3\,0})$. Consequently, we can pass to the limit and conclude that

$$
\sup_{0\le t\le T}\int (f_3u_{xxx}^2+f_2u_{xxy}^2+f_1u_{xyy}^2+u_{yyy}^2)+3\int_0^T\int (g_3u_{xxxx}^2+g_2u_{xxxy}^2+g_1u_{xxyy}^2)\le C.
$$

The case: $j\ge 4$.

In this case, we take $f_{\alpha_1,\delta}$ approximating $f_{\alpha_1} \in W_{0\ \alpha_1\ 0}$. We apply ∂^α to (2.1), multiply by $f_{\alpha_1,\delta}\partial^\alpha u$ and integrate over $\mathbb{R}^2$. We need to get a bound on

$$\left|\int_0^T \int f_{\alpha_1,\delta}(\partial^\alpha u)\partial^\alpha(uu_x)\right|.$$

In Lemma 6.2 below, we prove that

$$\left|\int_0^t \int f_{\alpha,\delta}(\partial^\alpha u)\partial^\alpha(uu_x)\right| \leq C + C\sum_{\gamma_1+\gamma_2=j}\int_0^t \int f_{\gamma_1,\delta}(\partial^\gamma u)^2 \tag{6.6}$$

where C depends only on terms bounded in the previous step of the induction. Consequently, we have that

$$\sup_{0\leq t\leq T}\sum_{|\alpha|=j} f_{\alpha_1,\delta}(\partial^\alpha u)^2 + \sum_{|\alpha|=j}\int_0^T \int (f_{\alpha_1,\delta})_x(\partial^\alpha u_x)^2 \leq C, \tag{6.7}$$

where C does not depend on δ, but only on T and the norm of $\phi \in X^1(\mathbb{R}^2)\cap \widetilde{H}_x^j(W_{0\ j\ 0})$. Passing to the limit, we get the desired estimate, namely,

$$\sup_{0\leq t\leq T}\sum_{|\alpha|=j} f_{\alpha_1}(\partial^\alpha u)^2 + \sum_{|\alpha|=j}\int_0^T \int g_{\alpha_1}(\partial^\alpha u_x)^2 \leq C. \tag{6.8}$$

$\square$

Theorem 6.2 *Let $f_{\alpha_1,\delta}$ approximate $f_{\alpha_1} \in W_{0\ \alpha_1\ 0}$. Let $j = |\alpha|$, $4 \leq j \leq K$. The following inequality holds:*

$$\left|\int_0^t \int f_{\alpha_1,\delta}(\partial^\alpha u)\partial^\alpha(uu_x)\right| \leq C + C\sum_{|\gamma|=j}\int_0^t \int f_{\gamma_1,\delta}(\partial^\gamma u)^2 \tag{6.9}$$

for $0 \leq t \leq T$ where C depends only on

$$\sup_{0\leq t\leq T}\int f_{\gamma_1,\delta}(\partial^\gamma u)^2 \tag{6.10}$$

$$\int_0^T \int (f_{\gamma_1,\delta})_x(\partial^\gamma u_x)^2 \tag{6.11}$$

for $\gamma = (\gamma_1,\gamma_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $|\gamma| \leq j-1$.

Proof. In order to get bounds on the left-hand side of (6.9), we use the fact that every term in the integrand is of the form

$$f_{\alpha,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) \tag{6.12}$$

where $r_i + s_i = \alpha_i$. Before showing the bounds on each of the terms in the integrand we point one bound we will be using frequently:

$$||\partial^\gamma u||_{L^\infty} \leq C\left(\int (\partial^\gamma u)^2 + (\partial^\gamma u_{xx})^2 + (\partial^\gamma u_y)^2\right)^{1/2}. \tag{6.13}$$

The case $|s| = j$. In this case, $|r| = 0$ and $s = \alpha$. Therefore,

$$\begin{aligned}\int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) &= \int f_{\alpha_1,\delta} u(\partial^\alpha u)(\partial^\alpha u_x) \\ &= -\frac{1}{2}\int [f_{\alpha_1,\delta} u]_x (\partial^\alpha u)^2 \\ &\le C||u||_{X^0} \int f_{\alpha_1,\delta}(\partial^\alpha u)^2.\end{aligned}$$

The case $|s| = j - 1$. Therefore, $|r| = 1$. We have two subcases below:
(a) **The subcase $r = (1, 0)$.** In this case,

$$\begin{aligned}\int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) &= \int f_{\alpha_1,\delta}(\partial^\alpha u) u_x (\partial^\alpha u) \\ &\le C||u_x||_{L^\infty} \int f_{\alpha_1,\delta}(\partial^\alpha u)^2.\end{aligned}$$

(b) **The subcase $r = (0, 1)$.** In this case,

$$\begin{aligned}\int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) &= \int f_{\alpha_1,\delta}(\partial^\alpha u) u_y (\partial_x^{\alpha+1}\partial_y^{\beta-2} u) \\ &\le C||u_y||_{L^\infty} \left(\int f_{\alpha_1,\delta}(\partial^\alpha u)^2\right)^{1/2} \left(\int f_{\alpha_1,\delta}(\partial_x^{\alpha_1+1}\partial_y^{\alpha_2-1} u)^2\right)^{1/2} \\ &\le C||u||_{X^1} \left[\int f_{\alpha_1,\delta}(\partial^\alpha u)^2 + \int f_{\alpha+1,\delta}(\partial_x^{\alpha_1+1,\alpha_2-1} u)^2\right].\end{aligned}$$

The case $|s| = j - 2$. We consider three subcases below:
(a) **The subcase $r = (2, 0)$.** In this case

$$\begin{aligned}\int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) &= \int f_{\alpha_1,\delta}(\partial^\alpha u) u_{xx} (\partial_x^{\alpha_1-1}\partial_y^{\alpha_2} u) \\ &\le C||f_{1,\delta} u_{xx}||_{L^\infty} \left(\int f_{\alpha_1,\delta}(\partial^\alpha u)^2\right)^{1/2} \left(\int f_{\alpha_1-1,\delta}(\partial_x^{\alpha_1-1}\partial_y^{\alpha_2} u)^2\right)^{1/2}.\end{aligned}$$

Now

$$f_{1,\delta} u_{xx} \approx \begin{cases} x u_{xx} & x > 1 \\ u_{xx} & x < -1. \end{cases}$$

For $x < -1$, we use the fact that

$$\begin{aligned}||u_{xx}||_{L^\infty(A_{-1})} &\le C\left(\int_{A_{-1}} (u_{xx}^2 + u_{xxxx}^2 + u_{xxy}^2)\right)^{1/2} \\ &\le C||u||_{X^1}.\end{aligned}$$

For $x > 1$, we use the fact that

$$\begin{aligned}f_{1,\delta} u_{xx} &= (f_{1,\delta} u)_{xx} - (f_{1,\delta})_{xx} u - 2(f_{1,\delta})_x u_x \\ &= (f_{1,\delta} u)_{xx} - (f_{1,\delta})_{xx} u - 2((f_{1,\delta})_x u)_x + 2(f_{1,\delta})_{xx} u.\end{aligned}$$

Therefore,

$$
\begin{aligned}
||f_{1,\delta}u_{xx}||_{L^\infty} &\leq ||(f_{1,\delta}u)_{xx}||_{L^\infty} + C||((f_{1,\delta})_x u)_x||_{L^\infty} + C||(f_{1,\delta})_{xx}u||_{L^\infty} \\
&\leq C\left(\int ((f_{1,\delta}u)_{xx})^2 + ((f_{1,\delta}u)_{xxxx})^2 + ((f_{1,\delta}u)_{xxy})^2\right)^{1/2} \\
&+ C\left(\int (((f_{1,\delta})_x u)_x)^2 + (((f_{1,\delta})_x u)_{xxx})^2 + (((f_{1,\delta})_x u)_{xy})^2\right)^{1/2} \\
&+ C\left(\int ((f_{1,\delta})_{xx}u)^2 + (((f_{1,\delta})_{xx}u)_{xx})^2 + (((f_{1,\delta})_{xx}u)_y)^2\right)^{1/2} \\
&\leq C + C\int u^2 + u_x^2 + f_{1,\delta}u_{xx}^2 + u_{xxx}^2 + f_{1,\delta}u_{xxxx}^2 + u_y^2 + u_{xy}^2 + f_{1,\delta}u_{xxy}^2 \\
&\leq C + C\sum_{|\gamma|\leq j}\int f_{\gamma_1,\delta}(\partial^\gamma u)^2.
\end{aligned}
$$

(b) **The subcase $r = (1,1)$.** Then

$$
\begin{aligned}
\int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) &= \int f_{\alpha_1,\delta}(\partial^\alpha u)u_{xy}(\partial_x^{\alpha_1}\partial_y^{\alpha_2-1}u) \\
&\leq ||u_{xy}||_{L^\infty}\left(\int f_{\alpha_1,\delta}(\partial^\alpha u)^2\right)^{1/2}\left(\int f_{\alpha_1,\delta}(\partial_x^{\alpha_1}\partial_y^{\alpha_2-1}u)^2\right)^{1/2} \\
&\leq C||u||_{X^1}\left(\int f_{\alpha_1,\delta}(\partial^\alpha u)^2\right)^{1/2}\left(\int f_{\alpha,\delta}(\partial_x^{\alpha_1}\partial_y^{\alpha_2-1}u)^2\right)^{1/2} \\
&\leq C + C\sum_{|\gamma|\leq j}\int f_{\gamma_1,\delta}(\partial^\gamma u)^2
\end{aligned}
$$

where C depends only on the bounds in the statement of the theorem.

(c) **The subcase $r = (0,2)$.** Then

$$
\begin{aligned}
\int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) &= \int f_{\alpha_1,\delta}(\partial^\alpha u)u_{yy}(\partial_x^{\alpha_1+1}\partial_y^{\alpha_2-2}u) \\
&\leq ||u_{yy}||_{L^\infty}\left(\int f_{\alpha_1,\delta}(\partial^\alpha u)^2\right)^{1/2}\left(\int f_{\alpha_1,\delta}(\partial_x^{\alpha_1+1}\partial_y^{\alpha_2-2}u)^2\right)^{1/2} \\
&\leq C||u_{yy}||_{L^\infty}\left(\int f_{\alpha_1,\delta}(\partial^\alpha u)^2\right)^{1/2} \\
&\leq C\left(\int u_{yy}^2 + u_{xxyy}^2 + u_{xyy}^2\right)^{1/2}\left(\int f_{\alpha_1,\delta}(\partial^\alpha u)^2\right)^{1/2} \\
&\leq C + C\sum_{|\gamma|=j}\int f_{\gamma_1,\delta}(\partial^\gamma u)^2
\end{aligned}
$$

where C depends only on the bounds in the statement of the theorem.

The case $|s| = j-3$ for $j \geq 5$.

In this case we consider $x > 1$ and $x < -1$ separately. First, for $x < -1$, we have

$$\int_{A_{-1}} f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) \le \int \left(\int_{A_{-1}} (\partial^\alpha u)^2\right)^{1/2} \left(\int_{A_{-1}} (\partial^r u)^4\right)^{1/4} \left(\int_{A_{-1}} (\partial^s u_x)^4\right)^{1/4}.$$

Now

$$\left(\int_{A_{-1}} (\partial^r u)^4\right)^{1/4} \le \left(\int_{A_{-1}} (\partial^r u)^2 + (\partial^r u_x)^2 + (\partial^r u_y)^2\right)^{1/2}$$

and $|r| = 3$ implies each of these terms is bounded by C where C depends only on

$$\int f_{\gamma_1,\delta}(\partial^\gamma u)^2 \qquad \text{for } |\gamma| \le j-1$$

since $j \ge 5$ and each of these terms has derivatives of order ≤ 4. Also,

$$\left(\int_{A_{-1}} (\partial^s u_x)^4\right)^{1/4} \le \left(\int_{A_{-1}} (\partial^s u_x)^2 + (\partial^s u_{xx})^2 + (\partial^s u_{xy})^2\right)^{1/2}$$

and $|s| = j-3$. Therefore, each of these terms has order at most $j-1$ and thus bounded by

$$C \sum_{|\gamma| \le j-1} \int f_{\gamma_1,\delta}(\partial^\gamma u)^2.$$

Therefore, for $x < -1$, we have

$$\int_{A_{-1}} f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) \le C + C \int_{A_{-1}} f_{\alpha_1,\delta}(\partial^\alpha u)^2,$$

where C depends only on the terms in the statement of the theorem.

Now for $x > 1$, we have

$$\begin{aligned}\int_{A_1} f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) \le &\left(\int_{A_1} f_{\alpha_1,\delta}(\partial^\alpha u)^2\right)^{1/2} \left(\int_{A_1} (f_{r_1/2,\delta}\partial^r u)^4\right)^{1/4} \\ &\times \left(\int_{A_1} (f_{(s_1+1)/2,\delta}\partial^s u_x)^4\right)^{1/4}.\end{aligned}$$

since $r_1 + s_1 = \alpha_1$. Now

$$\begin{aligned}\left(\int_{A_1} (f_{r_1/2,\delta}\partial^r u)^4\right)^{1/4} &\le \left(\int_{A_1} (f_{r_1/2,\delta}\partial^r u)^2 + ((f_{r_1/2,\delta}\partial^r u)_x)^2 + ((f_{r_1/2,\delta}\partial^r u)_y)^2\right)^{1/2} \\ &\le C\left(\int_{A_1} f_{r_1,\delta}(\partial^r u)^2 + f_{r_1,\delta}(\partial^r u_x)^2 + f_{r_1,\delta}(\partial^r u_y)^2\right)^{1/2}.\end{aligned}$$

Since $|r| = 3$, each of the terms above has order at most four, and, therefore, is bounded by a constant C which depends only on

$$\int f_{\gamma_1,\delta}(\partial^\gamma u)^2 \qquad \text{for } |\gamma| \le 4 \le j-1,$$

since here we are assuming $j \geq 5$.

Similarly,

$$\left(\int (f_{(s_1+1)/2,\delta}(\partial^s u_x))^4\right)^{1/4}$$
$$\leq C\left(\int (f_{(s_1+1)/2,\delta}(\partial^s u_x))^2 + ((f_{(s_1+1)/2,\delta}(\partial^s u_x))_x)^2 + ((f_{(s_1+1)/2,\delta}(\partial^s u_x))_y)^2\right)^{1/2}$$
$$\leq C\left(\int f_{s_1+1,\delta}[(\partial^s u_x)^2 + (\partial^s u_{xx})^2 + (\partial^s u_{xy})^2]\right)^{1/2}.$$

Since $|s| = j-3$ each of these terms is of order at most $j-1$. Therefore, each of these terms is bounded by

$$C \sum_{|\gamma| \leq j-1} \int f_{\gamma_1,\delta}(\partial^\gamma u)^2.$$

The case $|s| = j - 3$ when $j = 4$.

In this case, $|s| = 1$. Therefore, either $s = (1,0)$ or $s = (0,1)$. For $s = (1,0)$, we have

$$\int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) = \int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)u_{xx}$$
$$\leq \left(\int f_{\alpha_1,\delta}(\partial^\alpha u)^2\right)^{1/2}\left(\int (f_{r_1/2,\delta}\partial^r u)^4\right)^{1/4}\left(\int (f_{1,\delta}u_{xx})^4\right)^{1/4}.$$

Now

$$\left(\int (f_{r_1/2,\delta}(\partial^r u))^4\right)^{1/4} \leq \left(\int f_{r_1,\delta}[(\partial^r u)^2 + (\partial^r u_x)^2 + (\partial^r u_y)^2]\right)^{1/2}.$$

We note that $|r| = 3$. Therefore, each of these terms is at most of order $4 = j$. Further,

$$\left(\int (f_{1,\delta}u_{xx})^4\right)^{1/4} \leq C\left(\int f_{2,\delta}[u_{xx}^2 + u_{xxx}^2 + u_{xxy}^2]\right)^{1/2}.$$

We note that each of these terms is of order at most $j-1$. Combining these estimates, we conclude that

$$\int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) \leq C + C\sum_{|\gamma|=j}\int f_{\gamma_1,\delta}(\partial^\gamma u)^2,$$

where C depends only on

$$\sum_{|\gamma| \leq j-1} \int f_{\gamma_1,\delta}(\partial^\gamma u)^2.$$

The case in which $s = (0,1)$ is handled similarly.

The case $|s| \leq j - 4$.

In this case, we bound the terms as follows:

$$\int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) \leq ||f_{(s_1+1)/2,\delta}\partial^s u_x||_{L^\infty} \left(\int f_{\alpha_1,\delta}(\partial^\alpha u)^2\right)^{1/2} \left(\int f_{r_1,\delta}(\partial^r u)^2\right)^{1/2}.$$

Now

$$||f_{(s_1+1)/2,\delta}(\partial^s u_x)||_{L^\infty} \leq C\left(\int f_{s_1+1,\delta}[(\partial^s u_x)^2 + (\partial^s u_{xxx})^2 + (\partial^s u_{xy})^2]\right)^{1/2}.$$

Since $|s| \leq j-4$, all of these terms are of order at most $j-1$. Therefore, each of these terms is bounded by

$$C \sum_{|\gamma|\leq j-1} \int f_{\gamma_1,\delta}(\partial^\gamma u)^2$$

and, therefore,

$$\int f_{\alpha_1,\delta}(\partial^\alpha u)(\partial^r u)(\partial^s u_x) \leq C + C\sum_{|\gamma|=j}\int f_{\gamma_1,\delta}(\partial^\gamma u)^2.$$

where C depends only on

$$C \sum_{|\gamma|\leq j-1} \int f_{\gamma_1,\delta}(\partial^\gamma u)^2.$$

□

7 Main Theorem

In this section we state and prove our main theorem, which states that if the initial data ϕ possesses certain regularity and sufficient decay at infinity, then the solution $u(t)$ will be smoother than ϕ. In particular if the initial data satisfies

$$\int \phi^2 + (1 + x_+^L)(\partial_x^L \phi)^2 + (\partial_y^L \phi)^2 < \infty,$$

then the solution will *gain* L derivatives in x. More specifically,

$$\int_0^T \int t^{L-1}(1 + e^{\sigma x_-})(\partial_x^{2L} u)^2 < \infty$$

for $\sigma > 0$ arbitrary, where T is the existence time of the solution.

Theorem 7.1 (Main Theorem). *Let $T > 0$ and let u be the solution of* (2.1) *in the region* $[0, T] \times \mathbb{R}^2$ *such that $u \in L^\infty([0, T] : \mathcal{Z}_L)$ for some $L \geq 2$. Then*

$$\sup_{0\leq t\leq T} \int_{\mathbb{R}^2} f_\alpha\,(\partial^\alpha u)^2\,dx\,dy + \int_0^T \int_{\mathbb{R}^2} g_\alpha\,(\partial^\alpha u_x)^2\,dx\,dy\,dt \leq C \tag{7.1}$$

for $L + 1 \leq |\alpha| \leq 2L - 1$, $2L - |\alpha| - \alpha_2 \geq 1$ where $f_\alpha \in W_{\sigma,\,2L-|\alpha|-\alpha_2,\,|\alpha|-L}$ and $g_\alpha \in W_{\sigma,\,2L-|\alpha|-\alpha_2-1,\,|\alpha|-L}$, $\sigma > 0$ arbitrary.

Proof. By assumption, $u \in L^\infty([0,\,T] \,:\, \mathcal{Z}_L)$. Therefore $u_t \in L^\infty([0,\,T] \,:\, L^2(\mathbb{R}^2))$, then $u \in C([0,\,T] \,:\, L^2(\mathbb{R}^2)) \cap C_w([0,\,T] \,:\, \mathcal{Z}_L)$. Hence $u : [0,\,T] \to \mathcal{Z}_L$ is a weakly continuous function. In particular, $u(\,\cdot\,,\,\cdot\,,t) \in \mathcal{Z}_L$ for every t. Let $t_0 \in (0,\,T)$ and $u(\,\cdot\,,\,\cdot\,,t_0) \in \mathcal{Z}_L$, then there are $\{\phi^{(n)}\} \subseteq C_0^\infty(\mathbb{R}^2)$ such that $\partial_x^{-1}\phi_{yy}^{(n)}$ are in $C_0^\infty(\mathbb{R}^2)$ and $\phi^{(n)}(\,\cdot\,,\,\cdot\,) \to u(\,\cdot\,,\,\cdot\,,t_0)$ in $\mathcal{Z}_L$. Let $u^{(n)}$ be the unique solution of (2.1) with initial data $\phi^{(n)}(x,\,y)$ at time $t = t_0$. By Corollary 4.4, the solution $u^{(n)} \in L^\infty([t_0,\,t_0+\delta] \,:\, X^1(\mathbb{R}^2))$ for a time interval δ which not depend on n. By Theorem 6.1, $u^{(n)} \in L^\infty([t_0,\,t_0+\delta] \,:\, \mathcal{Z}_L)$ and

$$\int_{t_0}^{t_0+\delta}\int_{\mathbb{R}^2} g_{\alpha_1}\,(\partial^\alpha u_x)^2\,dx\,dy\,dt \le C \tag{7.2}$$

for $|\alpha| = L$, $\alpha_1 \neq 0$, where $g_{\alpha_1} \in W_{\sigma,\,\alpha_1-1,\,0}$ and C depends only on the norm of $\phi^{(n)} \in \mathcal{Z}_L$. Also by Theorem 6.1, we have (non-uniform) bounds on

$$\sup_{[t_0,\,t_0+\delta]}\sup_{(x,y)}\left[(1+x^+)^k|\partial_x^{\alpha_1}u^{(n)}(x,\,y,\,t)| + |\partial_y^{\alpha_2}u^{(n)}(x,\,y,\,t)|\right] < +\infty \tag{7.3}$$

for each n, k and α_1, α_2. Therefore, the a priori estimates in Lemma 5.1, are justified for each $u^{(n)}$ in the interval $[t_0,\,t_0+\delta]$.

We start our induction with $|\alpha| = L+1$. In this case, we take $g_\alpha \in W_{\sigma,\,L-2-\alpha_2,\,1}$ and let $f_\alpha = \frac{1}{3}\int_{-\infty}^x g_\alpha(z,\,t)\,dz$. We note that $2L - |\alpha| - \alpha_2 \ge 1$ by assumption. Therefore, $L - 2 - \alpha_2 \ge 0$. As shown in Lemma 5.1, we have the following bounds on the higher derivatives of $u^{(n)}$,

$$\sup_{[t_0,\,t_0+\delta]}\int_{\mathbb{R}^2} f_\alpha\,(\partial^\alpha u^{(n)})^2\,dx\,dy + \int_{t_0}^{t_0+\delta}\int_{\mathbb{R}^2} g_\alpha\,(\partial^\alpha u_x^{(n)})^2\,dx\,dy\,dt \le C \tag{7.4}$$

where C depends only on the norm of $u^{(n)} \in L^\infty([0,\,T] \,:\, \mathcal{Z}_L)$ and the term in (7.2). We conclude, therefore, that the constant C in (7.4) depends only on $||\phi^{(n)}||_{\mathcal{Z}_L}$. We continue this procedure inductively. For the $|\alpha|^{th}$ step, let $g_\alpha \in W_{\sigma,\,2L-|\alpha|-\alpha_2-1,\,|\alpha|-L}$ for $\alpha_2 \le 2L-|\alpha|-1$ and define $f_\alpha = \frac{1}{3}\int_{-\infty}^x g_\alpha(z,\,t)\,dz$. The non-uniform bounds on $u^{(n)}$ in (7.2) allows us to use Lemma 5.1 and our inductive hypothesis to conclude that

$$\sup_{[t_0,\,t_0+\delta]}\int_{\mathbb{R}^2} f_\alpha\,(\partial^\alpha u^{(n)})^2\,dx\,dy + \int_{t_0}^{t_0+\delta}\int_{\mathbb{R}^2} g_\alpha\,(\partial^\alpha u_x^{(n)})^2\,dx\,dy\,dt \le C$$

where again C does not depend on n, but only on the norm of $\phi^{(n)} \in \mathcal{Z}_L$. By Corollary 4.5,

$$u^{(n)} \overset{*}{\rightharpoonup} u \quad \text{weakly in} \quad L^\infty([t_0,\,t_0+\delta] \,:\, X^1(\mathbb{R}^2)).$$

Therefore, we can pass to the limit and conclude that

$$\sup_{[t_0,\,t_0+\delta]}\int_{\mathbb{R}^2} f_\alpha\,(\partial^\alpha u)^2\,dx\,dy + \int_{t_0}^{t_0+\delta}\int_{\mathbb{R}^2} g_\alpha\,(\partial^\alpha u)^2\,dx\,dy\,dt \le C. \tag{7.5}$$

This proof is continued inductively up to $|\alpha| = 2L-1$. Since δ does not depend on n, this result is valid over the whole interval $[0,\,T]$. $\square$

References

[1] M.J. Ablowitz & P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.

[2] O. V. Besov, V. P. IL'IN and S. M. Nikolskii, *Integral representations of functions and imbedding theorems,* Vol. I, J. Wiley 1978.

[3] J.C. Ceballos, M. Sepulveda and O. Vera, *Gain in regularity for a coupled nonlinear Schrodinger system*, Boletim da Sociedade Paranaense de Matematica. To appear.

[4] A. Cohen, *Solutions of the Korteweg-deVries equation from irregular dat*, Duke Math. J. **45** (1978), 149-181.

[5] P. Constantin and J.C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. **1** (1988), 413-439.

[6] W. Craig & J. Goodman, *Linear dispersive equations of airy type*, J. Diff. Eqns. **7** (1990), 38-61.

[7] W. Craig, T. Kappeler, & W. Strauss, *Gain of regularity for equations of KdV type*, Ann. Inst. H. Poincare **9** (1992), 147-186.

[8] D. David, D. Levi & P. Winternitz, *Solitons in shallow seas of variable depth and in marine straits*, Stud. Appl. Math. **80** (1989), 1-23.

[9] J. Ginibre and G. Velo, *Proprietes de lissage et existence de solutions pour l'equation de Benjamin-Ono generalisee*, C.R. Acad. Sci. Paris **308** (1989), 309-314.

[10] N. Hayashi, K. Nakamitsu, and M. Tsutsumi, *On solutions of the initial value problem for the nonlinear Schrodinger equation in one space dimension*, Math. Z. **192** (1986), 637-650.

[11] N. Hayashi, K. Nakamitsu, and M. Tsutsumi, *On solutions of the initial value problem for the nonlinear Schrodinger equation*, J. Funct. Anal. **71** (1987), 218-245.

[12] N. Hayashi and T. Ozawa, *Smoothing effect for some Schrodinger equations*, J. Funct. Anal. **85** (1989), 307-348.

[13] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Adv. in Math. Suppl. Studies: Studies in Appl. Math. **8** (1983), 93-128.

[14] B.B. Kadomtsev & V.I. Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Sov. Phys. Dokl. **15** (1970), 539-541.

[15] C.E. Kenig, *On the local and global well-posedness theory for the KP-I equation*, Annales de l'Institut Henri Poincare. Annales: Analyse Non Lineaire/Nonlinear Analysis **21**, 6 (2004) 827-838.

[16] C. Kenig, G. Ponce, and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40**, No. 1 (1991), 33-69.

[17] S. Kruzhkov and A.V. Faminskii, *Generalized solutions of the Cauchy problem for the Korteweg-deVries equation*, Math. U.S.S.R. Sb. **48** (1984), 93-138.

[18] J. Levandosky, *Gain of regularity for the KP-II equation*, Indiana Univ. Math. J. **49** (2000), 353-403.

[19] J. Levandosky, *Smoothing properties of nonlinear dispersive equations in two spatial dimensions*, J. Differential Equations **175** (2001), 275-352.

[20] L. Molinet, J.C. Saut, N. Tzvetkov, *Global well-posedness for the KP-I equation on the background of a non-localized solution*, Communications in Mathematical Physics **272**, 3 (2007) 775-810.

[21] V.I. Petviashvili & V.V. Yan'kov, *Solitons and turbulence*, Rev. Plasma Phys., XIV (1989), 1-62.

[22] G. Ponce, *Regularity of solutions to nonlinear dispersive equations*, J. Differential Equations **78** (1989), 122-135.

[23] J.C. Saut, *Remarks on the generalized Kadomtsev-Petviashvili equations*, Ind. Univ. Math. J. **42** (1993), 1011-1026.

[24] P.M. Santini, *On the evolution of two-dimensional packets of water waves over an uneven bottom*, Lett. Nuovo Cim. **30** (1981), 236-240.

[25] S. Ukai, *Local solutions of the Kadomtsev-Petviashvili equation*, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **36** (1989), 193-209.

[26] O. Vera, *Gain of regularity for a generalized coupled system of nonlinear evolution dispersive equations type,* Ph. D. Thesis, UFRJ, Rio de Janeiro, Brazil(2001).

[27] O. Vera, *Gain of regularity for a Korteweg-de Vries-Kawahara type equation*, Electron. J. Differential Equations **71** (2004), 1-24.